3+1D topological orders with (only) an emergent fermion

Heidelberg-Munich-Vienna Seminar on Mathematical Physics

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Theorem [JF]: Up to nonunique isomorphism, $\exists$ exactly three 3+1D bosonic topological orders with only one nontrivial particle:

- $\mathbb{Z}_2$ gauge theory] the particle is a boson
- spin-$\mathbb{Z}_2$ gauge theory
- anomalous version of spin-$\mathbb{Z}_2$ gauge theory

} the particle is a fermion

These slides are available at

http://categorified.net/Oct26slides.pdf

Operational / Algebraic defn of "topological order"

Motivation: A 2+1D topological order is a (unitary) nondegenerate braided fusion 1-category.

↑ but I don't know how to define "unitary" higher categories.

Objects in this 1-cat are:

- Line operators (e.g. Wilson lines)
- (quasi) particle excitations

these mean the same thing: every line is the world-line of a "particle".

Note: this classification includes a nontrivial statement: to know a 2+1D top. order, you do not need to know the surface ops.

Why not? Thm: In the "fusion" case, all codim-1 ops are condensations of codim-≥ 2 ops. (Any dim.)

Fusion $\iff$ $\mathbb{1}$ is indecomposable $\iff$ local vacuum is nondegenerate.

Defn: A 3+1 D top. order is a $\underbrace{\text{nondeg. braided}}$ $\underbrace{\text{fusion 2-category}}_{\text{" 3-1}}$ $\mathcal{B}$

"Fusion 2-cats" due to Douglas-Reutter 2018

"fusion n-cat" due to JF 2020.

i.e. trivial "Müger centre".

$\mathrm{Obj}(\mathcal{B})$ = (quasi) string excitations = surface operators.

$\Omega\mathcal{B} := \mathrm{End}_{\mathcal{B}}(\mathbb{1})$ = (quasi) particles = lines.

$\Omega\mathcal{B}$ is automatically a symmetric fusion category.

Tannakian duality [Deligne]: Either $\Omega\mathcal{B} = \mathrm{Rep}(G)$

or $\Omega\mathcal{B} = \mathrm{Rep}(G, \varepsilon)$ where G is a finite gp

and $\varepsilon \in Z(G)$ has order = 2.

:= Super reps in which ε acts as $(-1)^f$.

Theorem [Lan-Kong-Wen 2018]: If $\mathcal{B}$ is a 3+1D top. order w/ $\Omega\mathcal{B} = \mathrm{Rep}(G)$, then $\mathcal{B}$ is a DW for G. i.e. $\mathcal{B} = Z(2\mathrm{Vec}^{\alpha}[G])$ for some $\alpha \in H^4(BG; \mathbb{C}^\times)$.

Pf: $\mathcal{O}(G) \in \mathrm{Rep}(G) = \Omega\mathcal{B}$ is condensible.

Condense it to produce:

$\mathcal{B}$  $\mathcal{B}/\!/\mathcal{O}(G)$ is a new 3+1D top order w/ $\Omega(\mathcal{B}/\!/\mathcal{O}(G)) = \mathrm{Vec}$.

↑ all lines condense on wall.
Syms of $\mathcal{O}(G) = G$ = surfaces on wall.

Then use:

Theorem [JF 20]: If $\mathcal{B}'$ is 3+1D top order w/ $\Omega\mathcal{B}' = \mathrm{Vec}$, then $\mathcal{B}' = 2\mathrm{Vec}$.

<u>Example:</u> "One nontrivial particle": If $\Omega\mathcal{B}$ has exactly one nonidentity simple object e, then $e^2 \simeq 1$ and

$$\Omega\mathcal{B} \simeq \underbrace{\mathrm{Rep}(\mathbb{Z}_2)} \quad \text{or} \quad \mathrm{Rep}(\mathbb{Z}_2, e) \simeq \mathrm{SVec}.$$

in this case, by LKW thm, $\mathcal{B} = \mathrm{DW}$ thy for $\mathbb{Z}_2$ — aka "3+1D Toric Code".

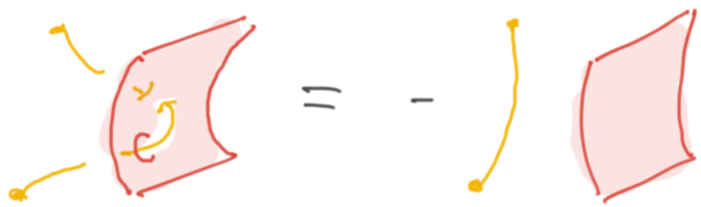
Since $H^4(B\mathbb{Z}_2; \mathbb{C}^\times) = 0$, it is unique.

e = Wilson line = charged particle = electron.

What else is there?

There is an "obvious" magnetic string $m \in \mathrm{Ob}(\mathcal{B})$.

It solves $m^2 \simeq \mathbb{1}$ and



Compare:

in 2+1D Toric Code

But wait, there's more!

$\mathcal{Z}(2\mathrm{Vec}[\mathbb{Z}_2]) \simeq \mathcal{Z}(2\mathrm{Rep}(\mathbb{Z}_2))$ is generated by e, m under: • linearization • direct sums • Karoubi completion

For example, $\underset{A}{\overset{\shortparallel}{\mathcal{O}}}(\mathbb{Z}_2) = 1 + e \in \Omega\mathcal{B}$ is an "idempotent" ↳ because $A^2 \xrightarrow{\text{multiplication}} A$

Its splitting, aka its <u>condensation</u>, is an indecomposable string $c \in \mathcal{B}$ called "Cheshire" by [Else-Nayak 17].

<u>Fusion rules:</u> $c^2 \simeq c \oplus c$.

Also, $\exists$ non-zero (noninvertible) 1-morphisms $\mathbb{1} \to c$, i.e. this string can "end".

Even though $\mathbb{1}, c$, are both simple!

<u>Defn:</u> Simple objects related by a 1-morphism are in the same <u>component</u>.

<u>Thm [Douglas Reutter 18]</u>

This is an equivalence relation.

"Schur's lemma for semisimple 2-categories."

What about when $\Omega\mathcal{B} = \mathrm{Rep}(\mathbb{Z}_2, e) = \mathrm{SVec}$, i.e. when e is a fermion?

Warning: Lan-Wen 18 claim a classification, but there is an error: They assert without proof that if $\Omega\mathcal{B} = \mathrm{SVec}$, then $\mathcal{B} = Z(2\mathrm{SVec}) = \text{Spin-}\mathbb{Z}_2$ gauge thy.
This assertion is wrong.

Correct classification: (1) Repeat LKW classification in the world of "supercategories", i.e. for $\mathcal{B} \boxtimes 2\mathrm{SVec}$.
(2) Conclude $\mathcal{B} \boxtimes 2\mathrm{SVec} \simeq$ super DW thy, classified by $SH^4(B\mathbb{Z}_2) = 0$!
(3) Study "Galois descent" along $2\mathrm{Vec} \hookrightarrow 2\mathrm{SVec}$.
Galois gp is $\mathbb{Z}_2^f[1]$ (i.e. classifying space is a $K(\mathbb{Z}_2, 2)$).

Conclusion: Two Gal actions: one nonanomalous and one anomalous. $\Rightarrow$ two options for $\mathcal{B}$.

I will write $\mathcal{Z}$ for the "nonanomalous" case, and $\mathcal{T}$ for the "anomalous" one. Let me describe them. By fiat, particle content is $\Omega\mathcal{Z} = \Omega\mathcal{T} = \mathrm{SVec} = \{1, e\}$.

↑ now a fermion.

So identity component is $2\mathrm{SVec}$.

Since $\mathrm{SVec} \simeq \mathrm{Rep}(\mathbb{Z}_2)$ monoidally, $2\mathrm{SVec} \simeq 2\mathrm{Rep}(\mathbb{Z}_2)$ at the level of semisimple 2-categories. In particular, identity component has two simple objects: $\mathbb{1}$ and c.

<u>Major difference</u>: c comes from condensing $1 \oplus e \simeq \mathrm{Cliff}(1)$, so $c \otimes c \simeq \mathrm{Cliff}(2) \simeq \mathbb{1}$. In particular, c is invertible.

<u>Note</u>: c has unusual statistics:

In both $\mathcal{S}$ and $\mathcal{T}$, there is also a magnetic component with simple objects m and $m' = cm$.

The characterizing property of magnetic strings is

In other words, m acts on $2\mathrm{SVec}$ (= identity cmp.) as the 1-form symmetry $\mathbb{Z}_2^f[1]$ generated by $(-1)^f$.

To build $\mathcal{S}$, $\mathcal{T}$, you adjoin m with this action.

$\mathcal{S} = 2\mathrm{SVec} \rtimes \mathbb{Z}_2^f[1]$. $\mathcal{T}$ = nontrivial extension.

Note: Both m and m' are order-2: $m^2 \simeq \mathbb{1} \simeq m'^2$.

Nontrivial fact: $\exists$ automorphism switching $m \leftrightarrow m'$.

In fact, $\pi_0 \mathrm{Aut}(\mathcal{S}) = \pi_0 \mathrm{Aut}(\mathcal{T}) = \mathbb{Z}_{16} = \{\text{MMEs of SVec}\}$. [Ising $\Updownarrow$ switch.]

If you come across $\mathcal{S}$ or $\mathcal{T}$ in the wild, how can you tell the difference?

Choose an iso $m^2 \simeq \mathbb{1}$. (There are two choices, related by an automorphism of $\mathcal{S}, \mathcal{T}$.)

This choice selects a braided monoidal sub-2-category with $\mathrm{ob} = \{\mathbb{1}, m\}$, 1-mor $= \{*\}$, 2-mor $= \mathbb{C}^\times$.

There are <u>two</u> braided monoidal 2-categories with these fusion rules, because

$$H^5(K(\mathbb{Z}_2, 2); \mathbb{C}^\times) \simeq \mathbb{Z}_2 .$$

The braiding structure is the <u>'t Hooft anomaly</u> of the $\mathbb{Z}_2[1]$-sym of $\mathcal{S}/\mathcal{T}$ generated by m.

What are these categories "physically"?

$\mathcal{J}$ = "spin-$\mathbb{Z}_2$ gauge theory", i.e. it is the 4D TQFT whose only fluctuating field is a <u>spin structure</u> η.

There is no sense for "$\eta = 0$", so no Dirichlet b.c., but there is a Neumann b.c., realizing $\mathcal{J} = Z(2\mathrm{SVec})$.

$\mathcal{T}$ = anomalous version with action $\int \frac{1}{2} \eta \wedge d\eta$.

It can be realized as a b.c. for the invertible 5D TQFT with partition fn $(-1)^{w_2 w_3}$.

<u>Thm (JF-Reutter, in progress)</u>: $\mathcal{T}$ is truly anomalous: it is not Morita-trivial, even among *framed* bosonic TQFTs.