In qft "orbifold" = gauge (a finite gp of automorphisms.)

High level heuristic picture:

locality type

Q is a "quantum field theory" property on spacetime

?D

$G \subseteq \mathrm{Aut}(Q)$. ← "global symmetries" "flavour"

In CFT context: spacetime $= \Sigma^2$, worldsheet

(0) Encode G-action on Q "locally".

$\rightsquigarrow$ Q to live on spacetimes + principal G-bundles.

"Couple Q to background G-connections"

← nondynamical

not gauging!

$S(\varphi) \rightsquigarrow S(\varphi, G\text{-connection})$

(1) "Dynamicalize" G-field.

↪ integrate over possible G-bundles

· can be obstructed; "anomaly"

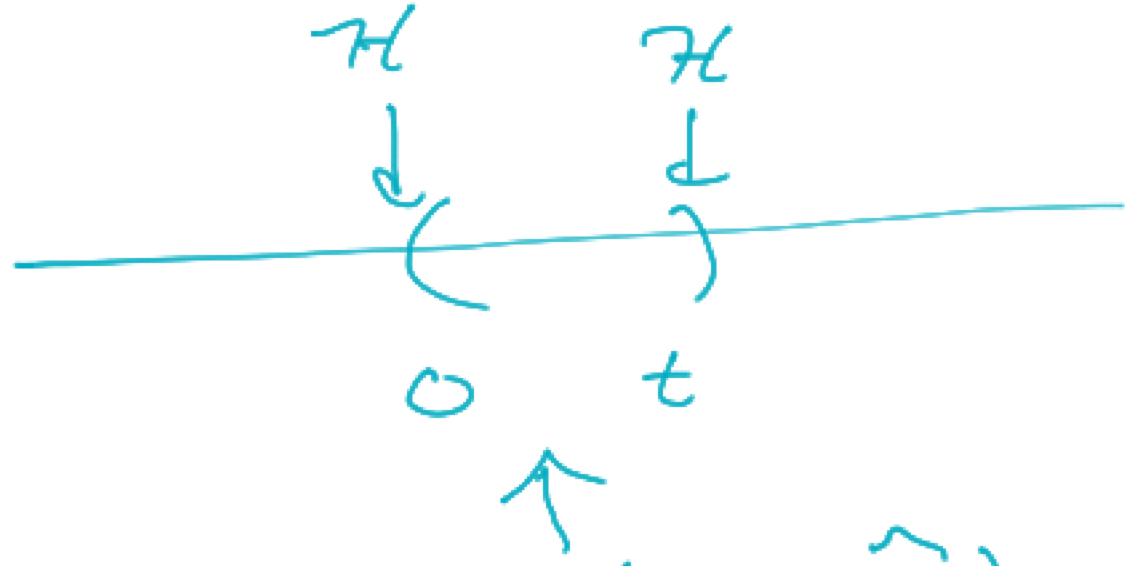
· choices ← "discrete torsion"

Warm up: 0+1 D QFT aka QM.

Schrödinger: A good model for a QM is $(\mathcal{H}, \hat{H})$

"Hilbert space" — $\mathcal{H}$

hamiltonian — $\hat{H}$



$\exp(it\hat{H})$

What is $\mathrm{Aut}((\mathcal{H}, \hat{H}))$?

might guess:
Stabilizer of $\hat{H}$
inside $\mathcal{U}(\mathcal{H})$ gp of unitary operators.

This guess is wrong.

among this data:
Factorization alg + translation invariance

$(\;) \mapsto \mathcal{B}(\mathcal{H})$.

A

I

$\exp(i\hat{H})A\exp(i\hat{H})$

$$\mathrm{Aut}\left((\mathcal{H}, \hat{H}) \right) \subseteq PU(\mathcal{H})$$

$$PU(\mathcal{H}) = \mathrm{Aut}(B(\mathcal{H}))$$

$g \in PU(\mathcal{H})$ s.t.

$g \hat{H} g^{-1} = \hat{H} + E$, $E \in \mathbb{R}$.

$$\begin{tikzcd} & U(\mathcal{H}) \arrow[d] \\ G \arrow[r, hook] \arrow[ur, dashed] & PU(\mathcal{H}) \end{tikzcd}$$

the anomaly will be the obstruction to this lifting.

discrete torsion will be the choice.

If I did have $G \hookrightarrow U(\mathcal{H})$ linear action then the orbifold theory would have Hilbert space

$\mathcal{H}/G$ (co)invariant.

An example: X some space $\circlearrowleft G$

QM of a particle moving in X "sigma model"

$$\mathcal{H} = L^2(X)$$

$$L^2(X/G) \approx L^2(X)/G.$$

Quantum field theorists like to calculate functions.

Wick-rotated partition function

"imaginary time $t = i\tau$"

$$Z(\tau) := \mathrm{Tr}_{\mathcal{H}}(\exp(-\tau \hat{H}))$$

$$\shortparallel$$

$$Z(\circlearrowleft_\tau = \mathbb{R}/\tau\mathbb{Z})$$

axiom: $\exp(-\tau \hat{H})$ traceclass for $\tau > 0$.

If $G \hookrightarrow U(\mathcal{H})$ "g-twined partition function"

(0)

$Z\Big($  $\Big) = Z_g(\tau) := \mathrm{Tr}_{\mathcal{H}}\big(g \exp(-\tau \hat{H}) \big)$

$\mathbb{R}/\tau\mathbb{Z}$

↑ this is the circle w/ circumference τ and the principal G-bundle with monodromy $g \in G$.

(1) $Z^{orb}(\tau) = \mathrm{Tr}(\mathcal{H}/G \, ; \, \dots)$ [Frobenius]

$$= \frac{1}{|G|} \sum_{g \in G} Z_g(\tau) = \sum_{[g] \in G/G} \frac{1}{|C_G(g)|} Z_{[g]}(\tau).$$

How to express $B(\mathcal{H}/G)$ in terms of G actn?

$A = B(\mathcal{H})$

★ <u>it</u> depends on more than just $G \circlearrowright A$.

Which algebras are $\approx$ ~~B(Hilbert~~ space)? Mat(f.d. vectr space)

<u>Answer</u>: the ~~von Neumann~~ assoc algs algebras which are

- morita invertible
- $\mathrm{Bimod}(A)$ is trivial.
- "Azumaya algebras"

$\mathcal{A}$ is $\underbrace{\text{Azumaya} = \text{central simple}}_{\text{“Heisenberg uncertainty”}}$

$$\begin{tikzcd} \mathbb{G}_m \arrow[d] & \\ \mathcal{A}^\times \arrow[d, two heads, "\text{Noether's thm}"'] & \text{form of } GL_n \\ \mathrm{Aut}(\mathcal{A}) & \text{form of } PGL_n \end{tikzcd}$$

quantum moment map.

$$\begin{tikzcd} & \mathcal{A}^\times \arrow[d] & \mathcal{A} \arrow[dd, bend left] \\ G \arrow[ur, dashed, "\mu"] \arrow[r] & \mathrm{Aut}(\mathcal{A}) & \\ \mathfrak{g} \arrow[urr, dashed, "\mu"] \arrow[r] & \mathrm{Der}(\mathcal{A}) & \end{tikzcd}$$

obstruction is class in $H^2(G;\, \mathbb{G}_m)$

$\mathcal{A}/\!/G$ = quantum Hamiltonian reduction.

$A /\!/ G := \mathrm{End}_A(A \otimes_{\mathbb{C}G} \mathbb{1}) = (A \otimes_{\mathbb{C}G} \mathbb{1})^G$ "

$\mathcal{C}^\infty(M)$

$A^{\mathcal{O}} \otimes_{\mathbb{C}G} \mathbb{1}^{\mu}$ $= \mathcal{C}^\infty(\mu^{-1}(0))$ $\dfrac{\mathcal{C}^\infty(\mu^{-1}(0))}{G}$

$= $ coinvariants of A

$\mathrm{Sym}(\mathfrak{g})$ as a right G-module.

$\mathcal{U}\mathfrak{g}$

$M \xrightarrow{\mu} \mathfrak{g}^*$

"moment map" has left A-action.

If G finite and A Azumaya

some alg obj.

$\mathrm{Bim}(A^G) \cong \mathrm{Bim}(\mathbb{C}^\alpha G) \ni \mathbb{C}$

alg obj.

If you've trivialized this α, same as an algebra map $\mathbb{C}^\alpha G \to \mathbb{C}$

fixed subalgebra for $G \subseteq \mathrm{Aut}(A)$

$\oplus$ For each $g \in G \subseteq \mathrm{Aut}(A) = A^\times / \mathbb{G}_m$

g line $\subseteq A$ living over g

"α-twisted gp alg"

- 1D QFT $\leftrightarrow$ Azumaya algs.
- $A^G \longrightarrow A//G$

  is not the orbifold theory.

If you discover that $Z_g(\tau)$ really do live in a nontrivial line you could diagnose an anomaly.

- How would a physicist detect an anomaly?

If G acting w/ projectivity $\alpha \in H^2(G; U(1))$

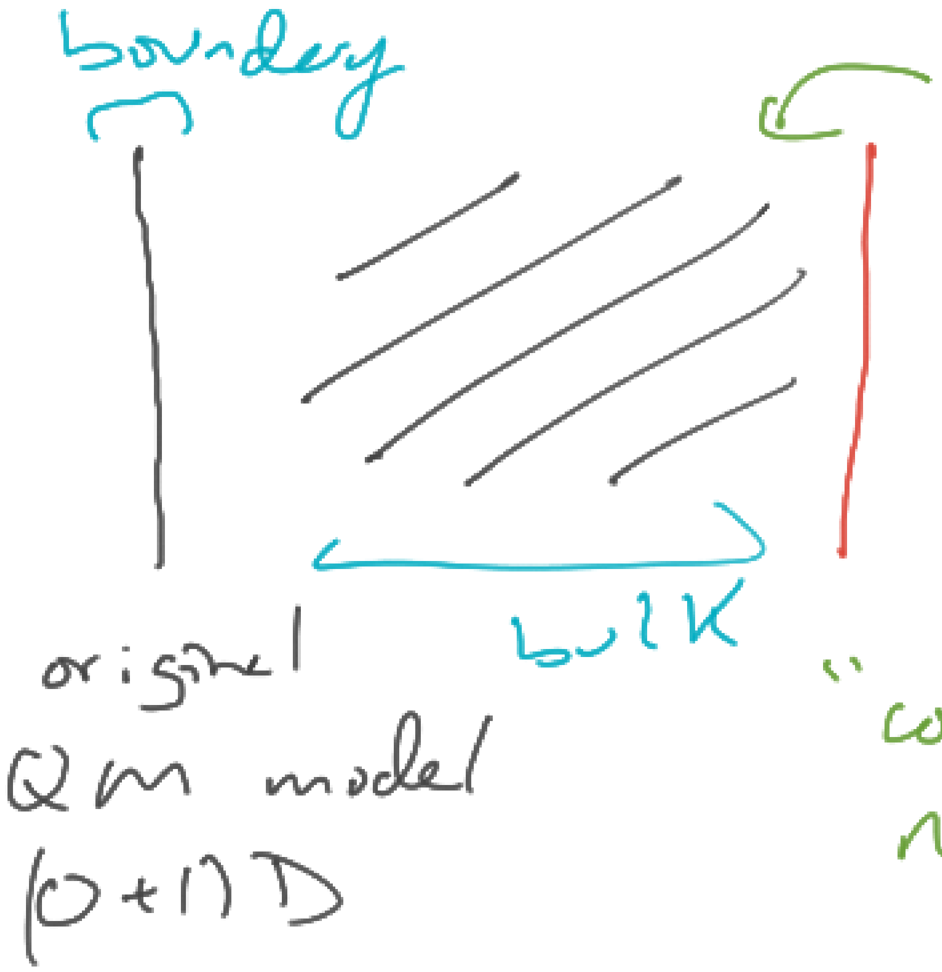
$Z_g(\tau) = Tr(g e^{(-\tau \hat{H})})$ is a meaningful elt of a line bundle on $LBG = \frac{G}{G}$ adjoint quotient.

line bundle on {circles w/ G-bundle}

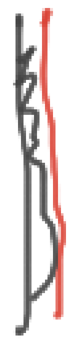
has a circumference.

$H^2(BG) \to H^1(LBG)$

has kernel for $G = \mathbb{Z}_2^2$ but inj for $\mathbb{Z}_3^2$.



flood w/
G gauge fields.



= orbifold thy.

boundary condition: field can be free at the boundary, but it should couple to boundary fields

"coupled neumann"

"anomaly inflow mechanism"

$H^2(G;U(1)) \ni \alpha$

if G acts on QM model w/ anomaly α, then bulk thy is pure G gauge thy w/ Dijkgraaf-Witten action α.

study other b.c. If $\alpha = 0$, then "pure neumann b.c."

1+1D = 2D QFT:

$$T_{ij} = \frac{\partial @}{\partial g^{ij}}$$

$\cup\!|$ no complete math defn.

2D CFTs almost defined

$$T_{ij}\, g^{ij} = 0.$$

$\cup\!|$

2D holomorphic CFTs.

$$T_{\bar z \bar z} = 0.$$

all operators depend only holomorphically on where they are inserted.

chiral alg VOA V $\equiv$ all operators. + niceness

/ insist that V be holomorphic $\to$ only one simple vertex module.

$\mathrm{Bim}(\mathcal{A}) = \mathrm{Vec}$ $\quad$ $\mathrm{Rep}(V) = \mathrm{Vec}.$

$V = V_L$ for some even lattice $L \subseteq \mathbb{R}^n$

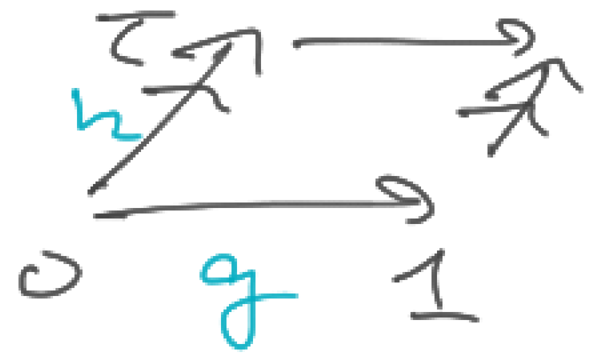
$\mathrm{Rep}(V) = L^*/L$ holomorphic $\iff$ L is unimodular.

Expect:

If V is nice holomorphic VOA and $G \subseteq \mathrm{Aut}(V)$ finite subgp

- might have an anomaly $\alpha \in H^3(G; U(1))$
- If we can trivialize α, then

$$V^G \longrightarrow V/\!/G$$

$E_\tau =$ 

- $$Z^{orb}(\tau) = \frac{1}{|G|} \sum_{(\pi_1 E_\tau, G)} Z_{g,h}(\tau).$$

• symptom of anomaly is nontriv. line bundles.

when I integrate over maps(E_τ, BG),

there's nothing to stop me from inserting

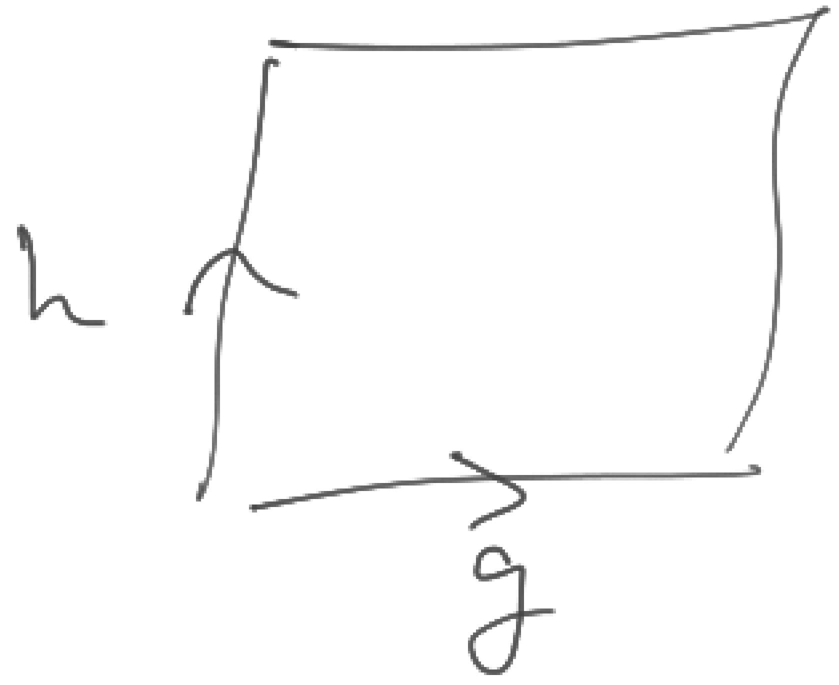
$$Z^{orb}(\tau) = \frac{1}{\#} \sum_{P \in maps(E_\tau, BG)} Z_{g,h}(\tau) \underbrace{\int_{E_\tau} P^* \beta}_{\text{extra DW term.}}$$

where $\beta \in H^2(BG; U(1))$

Choices of orb. thy = torsor for H^2

obstruction $\in H^3$.

$$Z^{orb}(\tau) = \sum_{[g]\in G/G} \underbrace{\frac{1}{|C(g)|}\sum_{h\in C_G(g)} Z_{g,h}(\tau)}_{=\ \text{Trace of something of } C(g)\text{-fixed points for some vector space depending on a choice } [g]\in G/G.}$$



g, h commute

$$V^G \to V/\!\!/G = \bigoplus_{[g]\in G/G} V(g)^{C(g)}$$

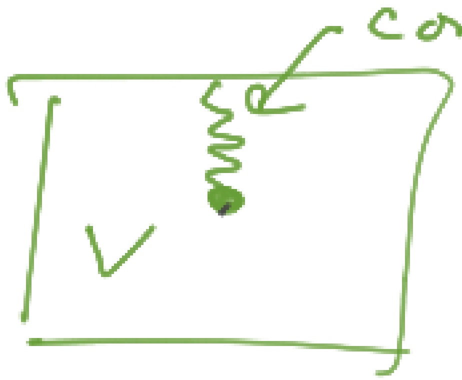
$V(g)$ is the hilbert space for S^1 w/ monodromy g.

i.e. $\text{order}(g) = n$

$$\mathbb{A}^1 \setminus 0 \longrightarrow \mathbb{A}^1 \setminus 0$$
$$z \longmapsto z^n$$

if I were a physicist, I'd draw in a branch cut

concentrate g on branch cut.



V descend it by using g.

Cat of extensions of this f-n. over puncture.

unpack: such an extension is almost a vertex module for V

i.e. $a \in V \rightsquigarrow$ a map

$$a(z) : M \longrightarrow M((z^{\pm 1}))$$

rather:

if $\quad g\,a = \exp(2\pi i \tfrac{k}{n})$

we ask for a map

$$a(z) : M \longrightarrow z^{k/n} M((z^{\pm 1})),$$

up to scalar.

If V is holo, then this cat has a unique simple object "$V(g)$". $V(g)$ will be a proj. $C(g)$-mod!

$G = \mathbb{Z}/n = \langle g \rangle$ expect anomaly $\in H^3(G; U(1))$
$\cong \mathbb{Z}/n$.

expect choices are torsor for $H^2(G; U(1)) = 0$.

If G acts w/ anomaly k, then $Z_{g,e}(\tau)$ is $\Gamma_0(n)$-modular where T^n acts w/ eigenvalue $\exp(2\pi i \frac{k}{n})$

$H^3(G; U(1))$

$\downarrow$ transgression

$S(\mathrm{Tr}(g\, q^{L_0 - c/24}; V)) = S(Z_{e,g}(\tau)) = Z_{g,e}(\tau)$

$H^1(\underbrace{\text{elliptic curves w/ } G\text{-bundle}/\mathbb{C}}; U(1))$

$$\pi_1\left(\text{stack} \ni \begin{tikzcd} \cdot \arrow[r] & \cdot \\ \cdot \arrow[u, "1"] \arrow[r, "g"'] & \cdot \arrow[u] \end{tikzcd}\right) = \Gamma_0(n) \times \mathbb{Z}/n$$

$$T[q^{\#}] = \exp(2\pi i \#) \cdot q.$$

T^n acts w/ eigenvalue $\frac{k}{n}$

$\Updownarrow$

$$Z_{g,e}(\tau) \in q^{k/n^2} \mathbb{C}[\![q^{\pm 1/n}]\!].$$

If $k = 0$

It is a thm [Moeller]

There is a unique way to promote these proj. actions to linear actions s.t.

$$V/\!/_{\mathbb{Z}/n} \overset{\text{want}}{=} \bigoplus_{g \in \mathbb{Z}/n} V(g)^{\mathbb{Z}/n}.$$

$\Leftarrow$ is a VOA.

$\mathrm{Aut}(V_{\text{leech}}) = T \cdot Co_0$

unique nonsplit extension. [Ivanov].

$T =$ leech torus.

Moellers thesis gives anomaly after restricting to any cyclic subgp.

not enough to tell a class in $H^4(T \cdot Co_0; \mathbb{Z})$

Somewhere I need V^G was nice.

Carnahan - Miyamoto that it actually is. (if G solv.)

in vN context: any finite gp (P. Xu)