Quantum Homotopy Types

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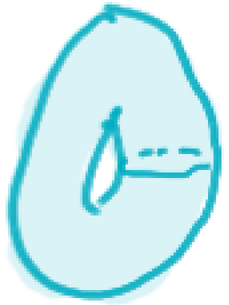
based on jt work in progress w/ David Reutter

Functorial Quantum Field Theory

wick-rotated

A $(d+1)$-dim'l quantum field theory supplies the following information:

- For each closed d-dim'l manifold N^d, a Hilbert space $\mathcal{H}(N^d)$.



Idea: $\mathcal{H}(N^d) = L^2(\text{field configurations on } N^d)$.

- For each $(d+1)$-dim'l cobordism M^{d+1}, a transition amplitude $Z(M): \mathcal{H}(\partial_{in} M) \to \mathcal{H}(\partial_{out} M)$.

Idea: $$\langle \varphi_{out} | Z(M) | \varphi_{in} \rangle = \int_{\substack{\text{fields } \varphi \text{ on } M \\ \text{s.t. } \varphi|_{\partial M} = \varphi_{in}, \varphi_{out}}} \exp\left(\int_M \mathcal{L}(\varphi) \right)$$

These data satisfy a composition law:

$$Z(M_2 \underset{N}{\circ} M_1) = Z(M_2) \underset{\mathcal{H}(N)}{\circ} Z(M_1)$$

Topological Quantum Field Theory

Warning: When a physicist says "manifold", it often means "manifold with a metric $g^{\mu\nu}$". So $\mathcal{H}(N)$, $Z(M)$ are really $\mathcal{H}(N, g_N)$, $Z(M, g_M)$.

Mathematicians usually mean "manifold w/o a chosen metric".

A QFT is topological if it is independent of $g^{\mu\nu}$.

Ur-example: Suppose X is a $\overbrace{\text{sufficiently finite}}^{\text{"}\pi\text{-finite"}}$ topological space.

Set

$$\mathcal{H}(N) = L^2(\text{homotopy classes of maps } N \to X)$$

$$Z(M) = \sum_{\substack{\text{homotopy classes of} \\ \text{maps } \varphi: M \to X \\ \text{w/ } \partial\varphi = \varphi_{in}, \varphi_{out}}} \frac{1}{|\mathrm{Aut}(\varphi)|}$$

compare Feynman diagram sum.

$\boxed{\text{"Homotopy sigma model"}}$

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"Homotopy sigma model"

Observation: These formulas make sense even if M, N have boundary.

In other words, the homotopy sigma model comes with a distinguished boundary condition, called the Neumann boundary.

Duality: Sometimes $X \not\cong Y$ but their sigma models are isomorphic. But duality permutes boundary conditions.

You can (mostly) recover X from the pair (σ-model for X, its Neumann b.c.).

mostly: if X is a "homotopy d-type".

The main idea of quantum homotopy theory is to treat any pair (TQFT, b.c.) as if it were a σ-model w/ Neumann b.c., but with some "quantum" target space.

Specifically, we want to do homotopy to these "quantum spaces". Extend homotopical invariants and constructions from classical spaces to quantum spaces.

$$\{\text{Classical spaces}\} \xhookrightarrow[\substack{(\sigma\text{-model,} \\ \text{N. b.c.})}]{} \{\text{quantum spaces}\}$$

Progress so far

The most basic homotopical invariants of a space X are its

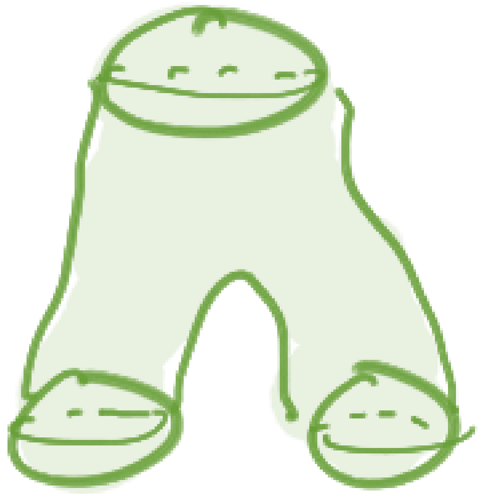
fundamental groupoid $\pi_{\leq 1} X$ and its

higher homotopy groups $\pi_k X$, $k = 2, 3, \ldots$

Technically, $\pi_k X$ is not a plain group, but a representation of $\pi_{\leq 1} X$.

(if $k \leq 2$)

Theorem: These can be recovered from the σ-model plus Neumann b.c. by looking at various spheres + pants.



Moreover, for any (TQFT, b.c.) in $\geq 3+1$ D, the same formulas work: you always get a groupoid "$\pi_{\leq 1} \mathcal{X}$" and a sequence of groups $\pi_2 \mathcal{X}$, $\pi_3 \mathcal{X}$, ...

In $< 3+1$ D, you get "quantum groups" aka Hopf algebras.

Quantum Phenomena

But you get more. You get a pairing $\pi_k \mathcal{X} \times \pi_{d-k} \mathcal{X} \to \mathbb{C}$, which is typically nontrivial, but is trivial for classical spaces. It measures "how quantum" your quantum space is.

Open Questions

- Detect classicalness. E.g. if the pairing is trivial, is $\mathcal{X}$ a classical sigma model?
- Understand quantum Postnikov data: how do the higher homotopy gps connect/talk to each other?
- Develop homology, Hurewicz theorem, etc.