Classification of TQFTs

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These slides available at
http://categorified.net/QMUL.pdf

Preamble and Punchline

My goal for this talk is to explain a quite remarkable classification of "all" TQFTs, except in 3D $= 2+1$D, where the classification is probably hopeless.

fully extended, etc., and in particular compact: defined on all closed spacetimes.

The strategy for the classification is due to Lan, Kong, and Wen, who explained a version in 4D.

Parts of my work are joint Hopkins, Reutter, and M. Yu.

To zeroth order, the classification says:

Every TQFT is a gauge theory for a finite higher group.

More precisely:

Every $(2m)$D TQFT is canonically a twisted generalized gauge theory for a finite $(m-1)$-group.

$n = 2m = 4$

1-gp = ordinary gp.

n = total spacetime dim

→ higher gp G with p-form symmetries for $p < m-1$ aka BG is a homotopy $(m-1)$-type.

→ the lagrangian is not just a Dijkgraaf-Witten action valued in $H^{2m}(BG; \mathbb{C}^\times)$, but rather in some generalized cohomology theory.

duality

p-form

$\updownarrow$

$n-p\pm\#$

→ e.g. a spin gauge theory, where $G = \mathbb{Z}_2^f \cdot G_b$ and instead of G-bundles $P \in H^1(M, G) = G$-valued 1-cocycles / gauge you use G-valued 1-cochains / gauge s.t. $\partial P = \omega_2$.

In $(2m+1)$D, marginally more complicated because of self-dual fields.

TQFTs

Defn: An compact and fully extended nD TQFT is a symmetric monoidal functor

$$Q : \mathrm{Bord}_n \longrightarrow \mathcal{V}^n,$$

I will use the "framed" version: the cobordism hypothesis says framed TQFTs are the algebraically simplest.

where:

Bord_n is the n-dim'l bordism n-category.

$M^n \longmapsto$ value of "path $\int$".

$M^{n-1} \longmapsto$ Hilbert space of states on M.

and:

$\mathcal{V}^n$ is ...:

← n-cat version of Vec.

(1) A categorical spectrum: $\Omega\mathcal{V}^n := \mathrm{End}_{\mathcal{V}^n}(\mathbb{1}) \simeq \mathcal{V}^{n-1}$

(2) $\mathbb{C}$-linear: $\mathcal{V}^0 = \Omega^n\mathcal{V}^n = \mathbb{C}$.

and usually $\mathcal{V}^1 = \mathrm{Vec}$.

(3) nice: basic linear algebraic constructions like $\oplus$s, images of idempotents, etc. are valid.

suffice to ask that $\mathcal{V}^n$ be a symmetric fusion n-cat.

You should think of the objects of $\mathcal{V}^k$ as "$\mathcal{V}$-enriched $(k-1)$-categories." The underlying $(k-1)$-category of $X \in \mathcal{V}^k$ is $\hom_{\mathcal{V}^k}(\mathbb{1}, X)$.

Main example: Given a TQFT Q, for $k = 0, \dots, n-1$

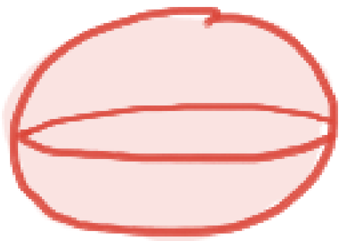
$$A^k_Q := Q(S_\partial^{n-1-k}) \in \mathcal{V}^{k+1}$$

↳ sphere with bounding framing

is the k-category of k-dimensional operators in Q.

The pair of pants makes A^k into a $(n-k)$-fold monoidal $(k-1)$-category.

and $A^{k-1} \simeq \Omega A^k := \mathrm{End}_{A^k}(\mathbb{1})$.



The strategy for classifying Q will be to study its operator content $\mathcal{A}^k$.

$\mathcal{A}^{n-1}$ is "nondegenerate" its centre is triv.

Example: Theorems of Schommer-Pries and Freed and Teleman imply that Q is invertible iff $\mathcal{A}^k$ is trivial for some (hence all) $k \geq \frac{n}{2} - 1$.

$n=4$

$\frac{n}{2}-1=1$

In other words, to detect invertibility, it's enough to look at

- 2D TQFT: $\mathcal{A}^0$ = local ops
- 3D TQFT: $\mathcal{A}^1$ = line ops
- 4D TQFT: $\mathcal{A}^1$ = line ops
- 5D TQFT: $\mathcal{A}^2$ = surface ops

etc.

Remark: This is the first hint that $n=2m$ has an easier classification than $n=2m+1$.

Recall that in general, A^k is $(n-k)$-fold monoidal. "E_{n-k}"

- 1-fold monoidal = associative.
- 2-fold = braided
- 3-fold = "syllepti c"

The Stabilization Hypothesis

says that, subject to some assumptions on $\mathcal{V}^n$, then

namely that it be "weak n" aka "(n,n)" and not "(∞,n)".

any ℓ-fold monoidal k-cat with $\ell \geq k+2$ is automatically symmetric monoidal.

fully commutative

Thus: A^k is symmetric if $k \leq \frac{n}{2} - 1$.

E.g: local ops when $n \geq 2$. line ops when $n \geq 4$. etc.

Compare with $k \geq \frac{n}{2} - 1$ for detecting invertibility.

An example of a symmetric monoidal K-category is

$$\mathrm{Rep}^{k}_{\mathcal{V}}(G) = \left\{ \begin{matrix} \text{representations of } G \\ \text{on objects in } \mathcal{V}^{k} \end{matrix} \right\}$$

where G is a k-group, i.e. $G = G_{(k-1)} \cdot G_{(k-2)} \cdot \ldots \cdot G_{(0)}$

and $G_{(p)}$ acts by p-form symmetries.

Note that there is a forgetful $\overset{\text{sym } \otimes}{\checkmark}$ functor $\mathrm{Rep}^{k}_{\mathcal{V}}(G) \to \mathcal{V}$.

You can reconstruct G as $\underset{\text{sym}\otimes}{\mathrm{Aut}}$(this functor).

Technical remarks:

(1) G should be "sufficiently small"

(2) In general, you reconstruct an "algebraic group scheme over $\mathcal{V}$".

Suppose that Q is a TQFT with $\mathcal{A}^k \simeq \mathrm{Rep}^k(G)$.

Then if G is finite you can condense aka ungauge to produce a new TQFT $Q /\!/ \mathrm{Rep}(G)$.

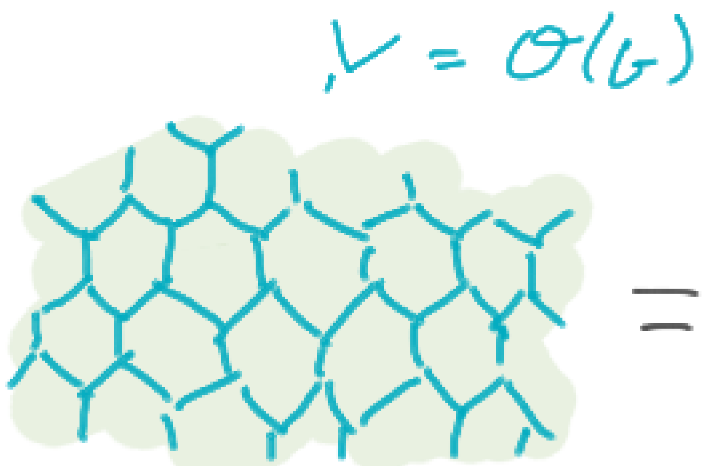
$\mathcal{O}(G)$ represents the forgetful functor

In practice, what you do is:

$\mathrm{Rep}^k(G)$ has a distinguished com. alg object

$$\mathcal{O}(G) = \{\mathcal{V}^{k-1}\text{-valued functions on } G\}.$$

Now flood Q by a network of $\mathcal{O}(G)$-defects, with junctions given by the alg. str.

$Q =$ $\qquad Q /\!/_{\mathrm{Rep}(G)} =$ 

$=$

The new "ungauged" $Q /\!/_{Rep(G)}$ comes with a nonanomalous G-action, since right and left actions of G on $\mathcal{O}(G)$ commute

and if you gauge this action, you get back Q.

OTOH, since $\mathcal{A}^{\kappa}_{Q} = Rep^{\kappa}(G)$, $\mathcal{A}^{\kappa}_{Q/\!/G}$ = trivial.

Suppose $n = 2m$ and $\underbrace{\kappa = m-1}$. Suppose $\mathcal{A}^{\kappa} = Rep^{\kappa}(G)$.

→ max dim for commutativity

Then, $Q /\!/_{Rep(G)}$ is invertible!

Thus, Q = invertible phase gauged by a G-action.

This data is classified by a generalized coh class in $(\mathcal{V}^{\times})^{n}(BG)$. $\mathcal{V}^{\times}$ is the spectrum of inv. TQFTs.

How often is a sym mon (higher) category the representations of a (higher) gp?

This is the subject of (higher) Tannakian duality.

The main step is to find a fibre functor $\mathcal{A}^k \to \mathcal{V}^k$ to replace $\mathrm{Forget}: \mathrm{Rep}^k_{\mathcal{V}}(G) \to \mathcal{V}^k$.

Defn: $\mathcal{V}^k$ is algebraically closed if every not-too-large sym. mon. $\mathcal{V}$-category admits a fibre.

↳ such size constraints are interesting but in any case automatic for $\mathcal{A}^k = \mathbb{Q}(S^{n-k-1})$ under some very mild assumptions about $\mathcal{V}$.

(0) d'Alembert, Gauss: $\mathbb{R}$ is not algebraically closed, but $\mathbb{C}$ is. Hilbert

(1) Deligne: $\mathrm{Vec}_{\mathbb{C}}$ is not algebraically closed, but $\mathrm{sVec}_{\mathbb{C}}$ is.

(2) JF - Hopkins: The 2-category of finite s.s. super-categories is algebraically closed.

(3) Freed - Scheimbauer - Teleman have constructed an alg. closed 3-category.

(n) JF - Reutter: we basically understand the alg. closed n-category $\forall n$, but work is still in progress.

(X) Hopkins: If $\mathcal{V}$ is alg. closed, then $\mathcal{V}^{\times} = I\mathbb{C}^{\times}$.

E.g.: In 4D or 6D, every super TQFT is a gauge thy.

What if $\mathcal{V}^\bullet$ is not alg. closed?

With Reutter, we understand the algebraic closure $\mathcal{V}^\bullet \hookrightarrow \mathcal{W}^\bullet$. It is a Galois extension: set $\mathrm{Gal} := \mathrm{Aut}_{\mathcal{V}}(\mathcal{W})$, then $\mathcal{V} = (\mathcal{W})^{\mathrm{Gal}}$.

$\Rightarrow$ TQFT over $\mathcal{V}$ = TQFT over $\mathcal{W}$ + Gal-equivariance

= gauge thy + twisting.

For $\mathcal{V}^k = \{\mathbb{C}\text{-lin } (k-1)\text{-cats}\}$, Gal is almost $SO(\infty)$, and is for small enough k.

$\Rightarrow$ In 4D or 6D, every bosonic TQFT is either a gauge thy or a spin gauge thy.

What about when $n = 2m+1$?

$Q' = Q/\!/_{\mathrm{Rep}(G)}$

Can condense out $\mathcal{A}^{m-1}$: There is an m-gp G s.t. $Q = Q'/\!/G$ and $\mathcal{A}'^{m-1} = \mathrm{triv}$.

$\mathcal{A}' =$ ops in Q'.

If $\mathcal{A}'^{m} = \mathrm{triv}$, then Q' is invertible and we are done.

What about in general? Then $\mathcal{A}'^{m}$ determines $\mathcal{A}'^{k}$ $\forall k$!

JF–Yu: Suppose $m \geq 2$. If $\mathcal{A}'^{-1} = \mathrm{triv}$, then $\mathcal{A}'^{m}$ is an abelian group!

symmetric when $m =$ odd
skew when $m =$ even

Moreover, it carries a nondegenerate $\underbrace{\text{(skew)}^{m-1}\text{symmetric}}$ pairing.

Moreover, if $\mathcal{V}$ is alg closed, so that $\mathcal{V}^{\times} = \mathbb{C}^{\times}$, then this pairing completely determines Q'.

Q + A

Hilbert's Nullstellensatz: ~1900

$\mathbb{K}[x, \dots, z] \,/\, \text{poly rel'n's}$

TFAE:

- $\mathbb{K}$ is ~~alg.~~ sep closed
- $\forall$ non-zero and small enough (finitely gen. / f.d. sep.) com $\mathbb{K}$-alg A

  $\exists$ com alg map $A \to \mathbb{K}$.

Deligne's existence of fibre functors ~2000

$\forall$ nonzero w/ size constraints sym $\otimes$ nice $\mathbb{K}$-lin cat $\mathcal{A}$

$\exists$ sym $\otimes$ functor $\mathcal{A} \longrightarrow sVec_{\mathbb{K}}$

if $\mathbb{K}$ alg closed field of char zero.

Thm– Lan Kong Wen:

lin ops = Rep(G)

A 4D TQFT w/ line ops = all bosons

is: $\Big(\underset{\text{finite gp}}{G}, \underbrace{H^4(BG; \mathbb{C}^\times), \text{ some small inv. phase}}_{} \Big)$

$\rightsquigarrow$ class in gen coh of BG w/ coefs in {bosonic inv. phases}.

if $G \neq G'$

line ops = Rep(G').

$(G', \quad \ldots)$

? It ~~can~~ cannot happen that $G \neq G'$ but Rep(G) and Rep(G') are equiv as sym fusion cats.

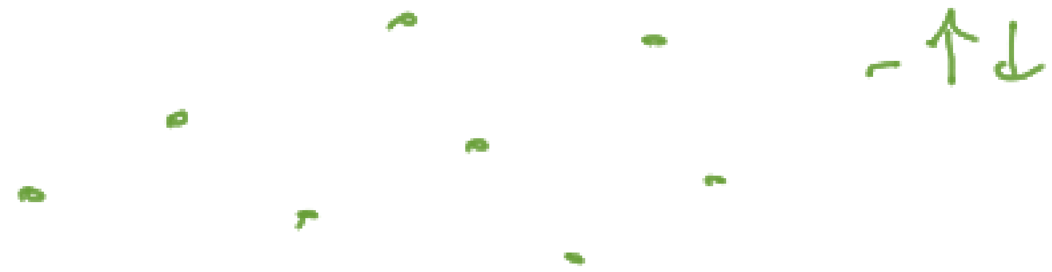
Preseminar discussion

Why classify TQFTs?

Mathematics answer: it is an interesting algebra problem.

Physics answer:
We really want to classify phases of quantum material.

E.g. of a material: metalic bar. QCD.



Place material on ∞-volume $\mathbb{R}^n$ $n+1$ D spacetime

(analysis): hope sensible Hilbert space.

Look at $\hat{H}$.

Condensed matter prejudice: min e-value of $\hat{H}$ is 0
this e-value appears w/ mult 1.

Typically expect spectrum of $\hat{H}$
- continuous
- e-values w/ ∞ mult, ...

$\swarrow$ $\searrow$

gapped
- 0 is the only e-value below some $\varepsilon > 0$.

insulating

gapless
- $\forall$ $(0, \varepsilon)$, $\exists$ infinite spectrum.

conducting

Assume: low energy behaviour of your material is modeled by effective ~~continuum~~ continuous QFT.

Fails for fractons.

Gapped phases are modeled by topological* QFT.

topological : $T_{\mu\nu} = 0$.

topological* : $T_{\mu\nu}$ is c-number.

microscopic: no lorentz inv.

macroscopic: small lorentz trans are ok, "large" ones ✗.

TQFT is allowed to couple to a framing.

a rep'n of lie alg
$\mathfrak{so}(n)$
might fail to be a rep of gp $SO(n)$.

$\hookrightarrow$ ∞-dim reps hopeless.

for f.d. reps, you could get $Spin(n)$ rep.

In top'l field thy, $\mathfrak{so}(n)$ ~~acts trivially.~~
action is trivialized.

but it can still happen that only Spin and not SO acts.

$$Spin(n) \longrightarrow SO(n) \longrightarrow \frac{SO(n)}{Spin(n)} = \mathbb{Z}_2(1)$$

1-form sym.

underlying space of $\mathbb{Z}_2(1)$
is $B\mathbb{Z}_2 = pt/\mathbb{Z}_2$

could act nontrivial on
top'l lines.