The fundamental theorem of deeper algebra

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these slides: https://categorified.net/QTCat.pdf

Plan for talk

- Higher algebra
- Deeper algebra
- Deeper Galois theory
- Deeper Kummer theory
- The cyclotomic character and a physics application

Higher algebra

The defining feature of sets is that any question you can ask about their elements has a yes/no answer.

- Is this element in that set?
- Are these two elements of that set equal?
- Are these two subsets of that set equal?

This rigid binary understanding of equality overlooks lots of "equality" type relationships.

- Are $\mathbb{Z}/3\mathbb{Z}$ and $\{x \in \mathbb{C} : x^3 = 1\}$ the same? Yes, they are isomorphic, in two different ways.

The defining feature of a groupoid is that questions about its elements have sets as their answers.

Higher algebra

The defining feature of a groupoid is that questions about its elements have sets as their answers. But why stop there?

The defining feature of a 2-groupoid is that questions about its elements have groupoids as their answers. But why stop there?

$$\operatorname{colim}^{\mathbf{Pr}}\big(\mathbf{Set} = \mathbf{0Gpd} \hookrightarrow \mathbf{1Gpd} \hookrightarrow \mathbf{2Gpd} \hookrightarrow \dots\big) =: \infty\mathbf{Gpd}$$

Definition: The defining feature of an ∞-groupoid is that questions about its elements of ∞-groupoids as their answers. ∞-groupoids are also called homotopy types and animae.

A good picture: elements of an ∞-groupoid are zero-dimensional things. Equalities between elements are one-dimensional. Equalities between equalities between elements are two-dimensional. So ∞-groupoids have higher-dimensional data.

Higher algebra

Definition: Higher (commutative) algebra is the (commutative) algebra that you get by using ∞-groupoids instead of sets.

Higher commutative algebra $\equiv E_\infty$-ring spectra aka multiplicative cohomology theories. Ordinary algebra $\equiv$ ordinary cohomology.

Higher commutative algebra $\supset$ ordinary commutative algebra: those E_∞-rings determined by their zero-dimensional data.

Deeper algebra

There is a different way to embed commutative algebra into a multi-dimensional world.

Given an ordinary commutative ring R, consider the category $\hat{\mathrm{B}}R :=$ {finitely-generated projective R-modules}. It is "additive" via $\oplus$ and "multiplicative" via $\otimes_R$: it is a 1-categorical ring. And:

$$\hom_{+,\times}(R,S) = \hom_{\oplus,\otimes}(\hat{\mathrm{B}}R, \hat{\mathrm{B}}S)$$

Definition: A 1-(categorical) ring is a symmetric monoidal additive and idempotent-complete 1-category.

Example: $\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(G)$. $\hom_{+,\times,\mathbb{C}}(\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(G), \hat{\mathrm{B}}\mathbb{C}) = \mathrm{B}G$.

Deeper algebra

Definition: A 1-(categorical) ring is a symmetric monoidal additive and idempotent-complete 1-category. But why stop there?

Definition: A 2-ring is a symmetric monoidal additive and higher-idempotent-complete 2-category. There is an embedding $\hat{\mathrm{B}} : \mathbf{1Ring} \hookrightarrow \mathbf{2Ring}$ sending $R \mapsto \{$fg proj R-module categories$\}$.

Example: $\hat{\mathrm{B}}\mathbf{Rep}^{\mathrm{fd}}_{\mathbb{C}}(G) = \mathbf{2Rep}^{\mathrm{fd}}_{\mathbb{C}}(G) = \{$finite ss G-module cats$\}$.

But why stop there?

$$\mathrm{colim}^{\mathbf{Pr}}\big(\mathbf{Ring} = \mathbf{0Ring} \overset{\hat{\mathrm{B}}}{\hookrightarrow} \mathbf{1Ring} \overset{\hat{\mathrm{B}}}{\hookrightarrow} \mathbf{2Ring} \overset{\hat{\mathrm{B}}}{\hookrightarrow} \dots\big) =: \infty\mathbf{Ring}$$

∞-rings are also called anticategories and categorical spectra

Deeper algebra

A good picture: R lives inside $\hat{\mathrm{B}}R$ at the top. Conversely, any n-ring S has an $(n-1)$-ring $\Omega S := \mathrm{End}_S(1_S)$. The extra data in S lives "below" the data of ΩS. Elements of a 0-ring are zero-dimensional things. So objects of an n-ring are $(-n)$-dimensional. So ∞-rings have deeper-dimensional data.

Deeper commutative algebra $\overset{\hat{\mathrm{B}}^\infty}{\supset}$ ordinary commutative algebra: those ∞-rings determined by their zero-dimensional data.

A deeper ring is a Cauchy-complete additive categorical spectrum. Replace "Cauchy-comp" with "presentable" to get Stefanich rings.

In an E_∞-ring spectrum, the spectrum structure is addition and the multiplication is extra. In an ∞-ring, and spectrum structure is multiplication and the addition is direct sums (limits and colimits).

Deeper Galois theory

Definition: For a finite ∞-group G, a G-Galois extension of an ∞-ring R is an ∞-ring S with a G-action and an isom $R \xrightarrow{\sim} S^G$.

The Hopf-Galois condition $S \otimes_R S \cong \mathrm{Fun}(G, S)$ is automatic.

Henceforth $R \in \infty\mathbf{Ring}_{\hat{\mathrm{B}}\infty\mathbb{Q}/}$, i.e. R has characteristic zero.

Proposition: $\{G\text{-Galois extensions of } R\} \cong \hom_{R/\infty\mathbf{Ring}}(R^G, R)$.

Proposition: There is a profinite ∞-groupoid $\text{ét}(R)$ such that $\{G\text{-Galois extensions of } R\} = \hom_{\infty\mathbf{Gpd}}(\text{ét}(R), \mathrm{B}G)$.

Definition: An ∞-ring is separably closed when $\text{ét}(R) = \mathrm{pt}$, i.e. when all Galois extensions are trivial.

Theorem: Each ∞-ring R has a unique separable closure $\overline{R}$: a separably closed (ind-)Galois extension.

Deeper Galois theory

Cardano: $\mathbb{R}$ is not separably closed. There is a nontrivial C_2-extension $\mathbb{R} \to \mathbb{C} = \mathbb{R}[x]/(x^2 = -1)$.

Argand, Gauss: $\mathbb{C}$ is separably closed among 0-rings.

Grassmann, Fermi, Dirac: $\hat{\mathrm{B}}\mathbb{C} = \mathbf{Vec}_{\mathbb{C}}^{\mathrm{fd}}$ is not separably closed. There is a nontrivial $\mathrm{B}C_2$-extension

$$\mathbf{Vec}_{\mathbb{C}}^{\mathrm{fd}} \to \mathbf{sVec}_{\mathbb{C}}^{\mathrm{fd}} = \mathbf{Vec}_{\mathbb{C}}^{\mathrm{fd}}[x]/(x^2 = \mathbb{C},\ \underset{x\ x}{\times} = -1).$$

Doplicher, Roberts, Deligne: $\mathbf{sVec}_{\mathbb{C}}^{\mathrm{fd}}$ is sep closed among 1-rings.

Hopkins, Yu: $\hat{\mathrm{B}}\mathbf{sVec}_{\mathbb{C}}^{\mathrm{fd}}$ is sep closed among 2-rings.

Freed, Scheimbauer, Teleman: $\hat{\mathrm{B}}^2\mathbf{sVec}_{\mathbb{C}}^{\mathrm{fd}}$ is not separably closed among 3-rings.

What does $\overline{\hat{\mathrm{B}}^\infty\mathbb{R}}$ look like? What is $\mathrm{Gal}(\overline{\hat{\mathrm{B}}^\infty\mathbb{R}}/\hat{\mathrm{B}}^\infty\mathbb{R})$?

Deeper Kummer theory

Suppose that R is an ordinary field of characteristic zero. If R is ordinary-separably-closed, then $\mu(R) := \mathbb{G}_m(R)_{\mathrm{torsion}} \cong \mathbb{Q}/\mathbb{Z}$.

Ordinary Kummer theory: If $\mu(R) \cong \mathbb{Q}/\mathbb{Z}$, then abelian (and hence solvable) extensions of R can be classified cohomologically.

Definition: A spectrum is an abelian group in the world of higher algebra. The thing that acts is not $\mathbb{Z}$ but $\mathbb{S}$ (for "stable sphere"). If A is a spectrum, there is an action map $\pi_n\mathbb{S} \times \pi_{-n}A \to \pi_0 A$. The dual to spheres is the unique spectrum $\mathrm{I}_{\mathbb{Q}/\mathbb{Z}}$ such that $\pi_n\mathbb{S} \times \pi_{-n}\mathrm{I}_{\mathbb{Q}/\mathbb{Z}} \to \pi_0 A = \mathbb{Q}/\mathbb{Z}$ is a perfect pairing.

Proposition: Suppose that R is an ∞-ring of characteristic zero. If R is separably closed, then $\mu(R) := \mathbb{G}_m(R)_{\mathrm{torsion}} \cong \mathrm{I}_{\mathbb{Q}/\mathbb{Z}}$.

Deeper Kummer theory

Proposition: Suppose that R is an ∞-ring of characteristic zero. If R is separably closed, then $\mu(R) := \mathbb{G}_m(R)_{\mathrm{torsion}} \cong \mathrm{I}_{\mathbb{Q}/\mathbb{Z}}$.

Theorem: Suppose that R is an ∞-ring of characteristic zero. Then $\mu(R) \cong \mathrm{I}_{\mathbb{Q}/\mathbb{Z}}$ if and only if R has no nontrivial abelian (and hence solvable) extensions.

Corollary: If $\mu(R) \cong \mathrm{I}_{\mathbb{Q}/\mathbb{Z}}$ and $\pi_1\,\text{ét}(R) = \mathrm{pt}$, then R is separably closed. If $\Omega^{\infty-1}R = \mathbf{sVec}_{\mathbb{C}}^{\mathrm{fd}}$, then $\pi_1\,\text{ét}(R) = \mathrm{pt}$.

Corollary: There is a unique separable extension R of $\hat{\mathrm{B}}^{\infty}\mathbb{R}$ with $\mu(R) \cong \mathrm{I}_{\mathbb{Q}/\mathbb{Z}}$ and $\Omega^{\infty-1}R = \mathbf{sVec}_{\mathbb{C}}^{\mathrm{fd}}$.

Hopkins: These conditions should characterize the universal category of topologically gapped phases of matter.

The cyclotomic character and a physics application

In ordinary Galois theory, the cyclotomic character is the map $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \mathrm{GL}_1(\widehat{\mathbb{Z}})$. In deeper Galois theory, it is the map $\mathrm{cyc} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(\mathrm{I}_{\mathbb{Q}/\mathbb{Z}}) = \mathrm{GL}_1(\widehat{\mathbb{S}})$.

Proposition: The homotopy groups of $\mathrm{Gal}(\overline{\hat{\mathrm{B}}^\infty\mathbb{R}}/\hat{\mathrm{B}}^\infty\mathbb{R})$ count topologically gapped phases not related by a topologically gapped interface. **Corollary:** The homotopy groups of $\mathrm{fib}(\mathrm{cyc})$ count invertible phases not admitting a topologically gapped boundary condition.

Theorem: $\mathrm{fib}(\mathrm{cyc})$ admits an L-theory description.
Proof strategy: Ideas from PL surgery. Ideas from "ungauging global categorical symmetries."

The cyclotomic character and a physics application

Kitaev: Invertible phases of matter form a spectrum with $\pi_0 = \mathbb{C}^\times$.

Corollary: If $\mathcal{I}$ is an invertible phase of n-dimensional matter whose induced map $\pi_n\mathbb{S} \to \mathbb{C}^\times$ is nontrivial, then either:

1. the induced map $\pi_n\mathbb{S} \to \mathbb{C}^\times$ is one of the six nontrivial Arf–Kervaire invariants, or:
2. $\mathcal{I}$ does not admit any topologically gapped boundary phases.

Gapped phases $\equiv$ insulators. Gapless phases $\equiv$ conductors.

Boundary phases for invertible phases $\equiv$ anomalous phases. This is a Lieb–Schultz–Mattis type theorem: anomalies force gaplessness. Except usual LSM-type theorems allow spontaneous symmetry breaking as a third case. Our theorem is stronger.


THANK YOU!