Quantum Homotopy Groups

Higher Categorical Tools for Quantum Phases of Matter

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Based on joint work in progress with David Reutter

these slides: http://categorified.net/QuantumHomotopy.pdf

Classical homotopy groups:

Any space $X \rightsquigarrow$ • groupoid $\pi_{\leq 1} X$.

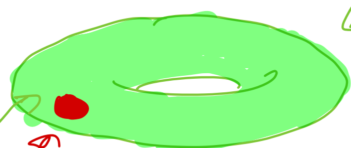
encodes the action of π_1 on π_k.

• $\pi_2 X, \pi_3 X, \ldots : \pi_{\leq 1} X \to \mathrm{AbGp}, \quad x \mapsto \pi_k(X, x)$

if X is sufficiently finite

$n+1$D sigma model w/ nD Neumann boundary

The sigma model "knows" the homotopy gps:

$\mathcal{H}\Bigg($  $\Bigg) = \mathbb{C}[\pi_k(X,x)]$ as a Hopf alg:

$S^k \times D^{n-k}$.

Dirichlet b.c. for $x \in X$

Neumann b.c. elsewhere

$m = \mathrm{Pants}^{k+1} \times D^{n-k}$

$\Delta = S^k \times \mathrm{cochaps}^{n-k+1}$

Goal: • Think of every $n+1$D TQFT w/ b.c. as a "sigma model w/ Neumann b.c" for some "quantum space"

• extract homotopy gps of this "quantum space".

The quantum fundamental gpoid

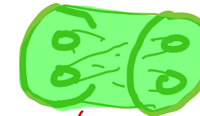
framing is ok!

Suppose Q is a (at least) once-extended open-close $n+1$D TQFT

The 1-category $Q(D^{n-1})$ is symmetric monoidal if $n \geq 3$.

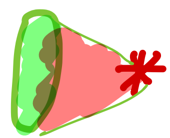
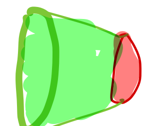
S^{n-2}

$: Q(D^{n-1}) \times Q(D^{n-1}) \xrightarrow{\otimes} Q(D^{n-1})$

braiding = 

Tannakian Philosophy: Every symmetric monoidal category $(\mathcal{C}, \otimes)$ should be thought of as $\mathrm{Rep}(\mathcal{G})$ for some gpoid $\mathcal{G} =$ "$\mathrm{Spec}\,\mathcal{C}$".

$\{\text{Points of } \mathcal{G}\} = \{\text{fibre functors } \mathcal{C} \longrightarrow \mathrm{Vec}\}$.

Think of these as choices  or maybe  "dir b.c." "neum b.c."

$\mathcal{C} \longrightarrow \mathrm{Vec}$ $\qquad$ $\mathcal{C} \longrightarrow \mathrm{Vec}$

Motivating calculation: If Q is a sigma model with Neumann b.c., then $Q(D^{n-2}) = \mathrm{Rep}(\pi_{\leq 1}\text{ target space})$.

So in an arbitrary open-closed TQFT, define $\pi_{\leq 1} Q := \mathrm{Spec}\, Q(D^{n-2})$.

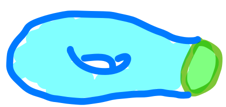
The quantum higher homotopy gps

Classically, $\pi_k X$ is an abelian gp w/ an action by $\pi_{\leq 1} X$.

Quantize: commutative and cocommutative Hopf alg internal to $\mathrm{Rep}(\pi_{\leq 1} X)$

So I'm after Hopf algebra objects internal to $\mathcal{Q}(D^{n-1})$.

Strategy: Take a solid n-manifold $(M^n, \partial M^{n-1})$. Take a "bite" out of the boundary. Apply $\mathcal{Q}$. This gives an object of $\mathcal{Q}(D^{n-1})$.

 : $\mathrm{Vec} = \mathcal{Q}(\varnothing) \longrightarrow \mathcal{Q}(D^{n-1})$

Theorem: $(S^k \times D^{n-k}) \setminus \text{bite}$ is a Hopf alg in D^{n-1}.

Multiplication $= (S^k \times \text{chaps}^{n+1-k}) \setminus \text{bite}$

chaps := solid pants.

Comultiplication $= (\text{pants}^{k+1} \times D^{n-k}) \setminus \text{bite}$

e.g.  $= \text{chaps}^2$.

"CPT thm": antipode $=$ (reflect one coord) $\times$ (reflect one coord), then untwist framings

The bite makes $S^k \times D^{n-k} \simeq S^k$ into a based sphere. $\text{pants}^{k+1} = \pi_k$ composition.

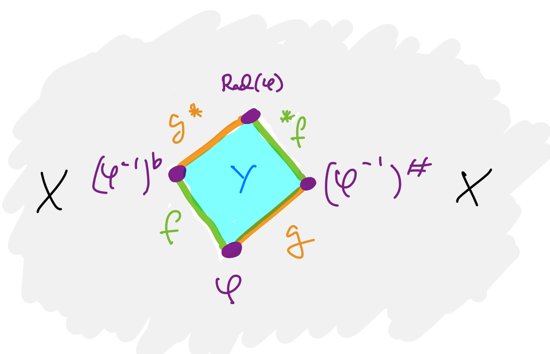
Constructing Hopf algebras

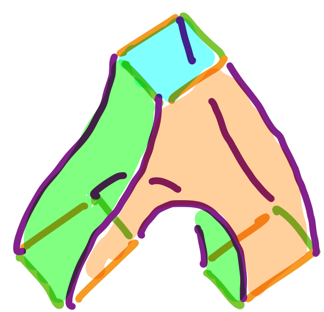
(see also Reutter's 2017 Perimeter talk)

The theorem takes place in a (once-extended) bordism category. So I am allowed to prove it in a more-extended bordism category.

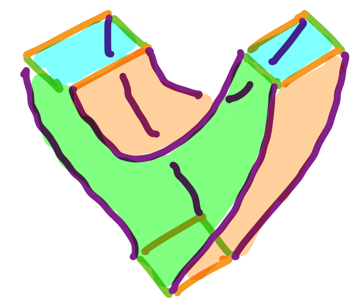
Corollary: Suppose $\mathcal{C}$ is an $(\infty,3)$-category w/ all adjoints. Given a (1-)retract $Y \underset{g}{\overset{f}{\rightleftarrows}} X$, $fg \overset{\varphi}{\cong} \mathrm{id}_X$, can get a Hopf alg $H(Y,f,g,\varphi)$ in the braided monoidal $(\infty,1)$-category $\mathrm{End}^{(2)}_{\mathcal{C}}(X) = \mathrm{End}_{\mathrm{End}_{\mathcal{C}}(X)}(\mathrm{id}_X)$.

Our pf is categorical/algebraic. Here is the topological interpretation:

$H :=$ 



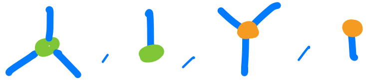
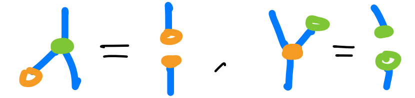
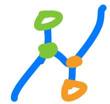
multiplication

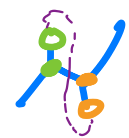


comultiplication

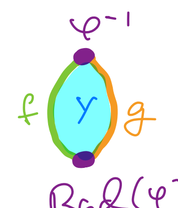
Interpretation: X = vac, Y = some QFT, f = Neumann b.c., g = Dirichlet b.c., φ = "Neum $\pitchfork$ Dir"

Twisted Frobenius – Hopf algebras

A bialgebra  is Frobenius-Hopf if it is equipped with an integral  and a cointegral , meaning  such that the composition  is an antiautomorphism. Our Hopf algebra looks Frobenius, but it isn't quite: the framings are wrong. Rather, it is **twisted Frobenius Hopf:** the integral and cointegral are (co)valued in some nontrivial invertible object $I =$

Facts: (1) Twisted Frobenius Hopf $\Rightarrow$ Hopf:  is the antipode.

(2) If $\mathcal{B}$ is rigid braided monoidal and Karoubi complete, the converse holds: every Hopf alg in $\mathcal{B}$ admits a unique twisted Frobenius Hopf str.

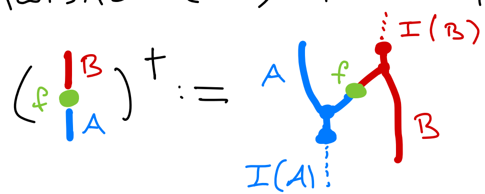
(3) In my example, the invertible is $I(Y, f, g, \varphi) =$ 

$\in \mathrm{End}^{(2)}(X)$. w/o "Rad", this would be $\mathrm{id}^{(2)}(X)$. "Rad" = "Radford" twists up the framings.

(Twisted) Frobenius-Hopf-Beck-Chevalley squares

A commuting square of adjunctible functors
$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "c"'] \arrow[dr, phantom, "\cong"] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r, "d"'] & \bullet \end{tikzcd}$$
is **Beck-Chevalley** if the vertical adjoint
$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \\ \bullet \arrow[u, "c^*"] \arrow[r, "d"'] & \bullet \arrow[u, "b^*"'] \end{tikzcd}$$
strongly commutes, where $(-)^*$ denotes the categorical adjoint.

A homomorphism between Frobenius(-Hopf) algebras doesn't have a categorical adjoint, but it does have a **linear adjoint** defined by the Frobenius pairings. In the twisted case, the adjoint is off by some invertible objects:



up to a suppressed power of the antipode

Definition: A square
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$
is (twisted) FHBC if there is an iso $I(A)I(D) \simeq I(B)I(C)$ s.t.
$$\begin{tikzcd} A\,I(A)^{-1}I(C) \arrow[r] \arrow[ddr, phantom, "\circ"] & B\,I(A)^{-1}I(C) \arrow[d, phantom, "\cong"] \\ & B\,I(B)I(D)^{-1} \\ C \arrow[uu] \arrow[r] & D \arrow[u] \end{tikzcd}$$

N.B. Iso is unique $\Leftarrow \mathrm{tr}(s^2)$ is invertible.
Such H is called *regular*.

(Twisted) Frobenius-Hopf exact sequences

Proposition: If $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence of finite gps which is exact at B, then the square of group algebras

$$\begin{tikzcd} \mathbb{K}A \arrow[r, "\mathbb{K}f"] \arrow[d] & \mathbb{K}B \arrow[d, "\mathbb{K}g"] \\ \mathbb{K} \arrow[r] & \mathbb{K}C \end{tikzcd}$$

is FHBC. The converse (FHBC $\Rightarrow$ middle-exact) holds if $|\mathrm{Ker}(g)| \neq 0$ in $\mathbb{K}$ (e.g. if $\mathbb{K}B$ is regular)

Definition: A sequence $\cdots \to A \to B \to C \to \cdots$ of Hopf algebras in a braided monoidal category should have

$$\begin{tikzcd} & B \arrow[dr] \arrow[dd, phantom, "\circ"] & \\ A \arrow[ur] \arrow[dr] & & C \\ & \mathbb{1} \arrow[ur] & \end{tikzcd}$$

at each entry.

A sequence of (twisted Frobenius) Hopf algebras is **Frob-Hopf exact** if these squares are FHBC.

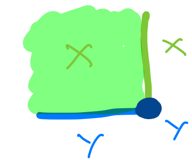
Example: If $A \xrightarrow{0} B \xrightarrow{0} C$ is FH exact at B and A and C are regular, then $B = \mathbb{1}$ is the trivial Hopf algebra.

FH exactness is not very strong in the irregular case.

The quantum Puppe sequence

Given a fibre bundle $\begin{matrix} F \to Y \\ \downarrow \\ X \end{matrix}$, get a LES of $\pi_{\leq 1} Y$-equivariant homotopy gps $\cdots \to \pi_k F \to \pi_k Y \to \pi_k X \to \pi_{k-1} F \to \cdots$

Quantum encoding: $X \rightsquigarrow$ sigma model w/ Neum b.c.



$\begin{matrix} Y \\ \downarrow \\ X \end{matrix} \rightsquigarrow$ another b.c., a corner

This is a relative open-closed TQFT.

Main Theorem: Every relative open-closed TQFT produces a FHLES of quantum homotopy gps.

Example: Calculate for

The Hopf algebras end up measuring fusion rings of observables in bulk + boundary. The differential measures the Hopf link.

Corollary: Bulk is invertible $\Longleftrightarrow$ boundary observables have nondegen. "higher S-matrix".