Some examples in fusion 2-categories and 3+1D topological order

Secret seminar, Nov 6, 2020. arXiv 2011.?????

Goal: Describe in detail the operator content of the three 3+1D topological orders ≡ bosonic anomalous TQFTs framed
with a unique nontrivial particle:

$\mathcal{R} := \mathbb{Z}_2$ gauge theory } particle is a boson

$\mathcal{S} :=$ Spin-$\mathbb{Z}_2$ gauge theory

$\mathcal{T} :=$ anomalous version of Spin-$\mathbb{Z}_2$ gauge theory. } particle is a fermion.

↳ Morally, the anomaly is $(-1)^{w_2 w_3}$.
I cannot (yet) prove that it is truly anomalous in the algebraic category.
With D. Reutter, can prove this anomaly is nontrivial in the (pseudo) unitary world.

Operational / algebraic defn of topological order:

Defn following [MS '89, RT '91, ...]: A 2+1D topological order is a (unitary) nondegenerate braided fusion 1-category $\mathbb{B}$.

↑, don't know what unitary higher categories

Physically, $ob(\mathbb{B})$ = line operators

= (quasi) particle excitations.

Thm (Folklore? in semisimple case Müger? David et al) non ss): nondegenerate $\Updownarrow$ $\mathbb{B}$ is invertible in $\underbrace{Mor_2(Cat)}_{4\text{-cat.}}$

Operational / algebraic defn of topological order:

Defn following [MS '89, RT '91, ...]: A ~~2~~ 3+1D topological order is a (unitary) nondegenerate braided fusion ~~1~~ 2-category $\mathcal{B}$.

Physically, $\mathrm{ob}(\Omega\mathcal{B})$ = line operators

= (quasi) particle excitations.

$\mathrm{ob}(\mathcal{B})$ = surface operators

= (quasi) string excitations.

$\Omega\mathcal{B} := \mathrm{End}_{\mathcal{B}}(\mathbb{1})$. Since $\mathcal{B}$ is braided, $\Omega\mathcal{B}$ is a sym. fusion 1-cat.

Defn [DR 18]: A fusion 2-category is a $\mathbb{C}$-linear finite semisimple 2-category, monoidal w/ all duals, and simple unit.

n-cat = weak n-cat = (n,n)-cat

A 2-category $\mathcal{C}$ is semisimple if it is:

2-cat = bicat

- locally semisimple: $\mathrm{hom}_{\mathcal{C}}(x,y)$ is a semisimple 1-cat.
- Karoubi complete.
- all 1-morphisms have adjoints.

← analog of being able to split all maps.

Main dif w/ 1-cat case: In 1-cats, Schur's Lemma says that a nonzero map between simple objects is an ISO.

This fails in 2-categories.

2-Categorical Schur's lemma [DR 18]:

If $f: X \to Y$ and $g: Y \to Z$ are nonzero maps between simple objects, then $g \circ f$ is nonzero.

$$\pi_0 \mathcal{C} := \{\text{simple objects in } \mathcal{C}\} / X \sim Y \text{ if } \exists\, \underset{\neq 0}{f} : X \to Y .$$

the components of $\mathcal{C}$.

s.s. 1-cat

s.s. 2-cat

$\mathcal{C}$ is finite s.s. if all $\hom_{\mathcal{C}}(X, Y)$ s are finite, and $\pi_0 \mathcal{C}$ is finite.

i.e. finitely many simple objects, finite dim hom spaces

Any simple object $X \in \mathcal{C}$ determines its component:

Define $\Omega_X \mathcal{C} := \mathrm{End}_{\mathcal{C}}(X)$. It is a fusion 1-cat.

Given a sep fusion 1-cat $\mathcal{F}$, define

$$\Sigma\mathcal{F} = \mathrm{Mod}^{\text{separable}}(\mathcal{F}) = \text{Karoubi completion of } B\mathcal{F}$$

Then $\Omega\Sigma\mathcal{F} \simeq \mathcal{F}$, and $\Sigma\Omega_X\mathcal{C}$ is the full subcat of $\mathcal{C}$ on $\oplus$s of objects in the comp. containing X.

<u>Note</u>: If X, Y in same component, then

$$\Omega_X\mathcal{C} \simeq \Omega_Y\mathcal{C}$$ <u>morita</u> <u>equiv</u> fusion cats, but <u>not</u> isomorphic.

Example time: $\mathbb{B}$ is nondeg braided fusion 2-cat.

Suppose $\Omega\mathbb{B}$ has unique nontriv. simple object "e".

Then $\underbrace{\Omega\mathbb{B}}_{\hookrightarrow \text{ sym.}} = \mathrm{Rep}(\mathbb{Z}_2)$ or SVec. $e^2 = 1$.

These are monoidally but not braidedly equivalent.

So identity component $\Sigma\Omega\mathbb{B} \subseteq \mathbb{B}$, as a linear 2-cat, (i.e. non-monoidal)
is indep. of whether e is a boson or fermion.

Two simple objects:

$$\mathbb{Z}_2 = \{1, e\} \circlearrowright \mathbb{1} \rightleftarrows c \circlearrowleft \mathbb{Z}_2 = \{id_c, e \cdot id_c\}$$

$\hom(\mathbb{1}, c) \cong \hom(c, \mathbb{1}) \cong \mathrm{Vec}$

$c :=$ "cheshire string" [EN '17]
$=$ unique alg. str. on $1 \oplus e$
$= \mathcal{O}(\mathbb{Z}_2)$ or $\mathrm{Cliff}(1)$.

The fusion rules depend on $\Omega\mathcal{B} \cong \mathrm{Rep}(\mathbb{Z}_2)$ or SVec:

so id comp. $\in \Omega\mathcal{B}$ two case are monoidally dif:

"R" S, T

$$c \otimes c \cong \begin{cases} c \oplus c, & \text{if } e \text{ is a boson,} \\ \mathbb{1}, & \text{if } e \text{ is a fermion.} \end{cases}$$

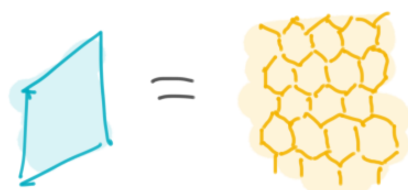
thy of a dynam. spin str.

<u>Fun exercise</u>: If e is a fermion, then

, i.e. $\beta_{c,c} \cong e \otimes \mathrm{id}_{c \otimes c}$.

Physically, c is built by "condensing an e particle".

The surface c looks microscopically like a network of e lines:



In spin-$\mathbb{Z}_2$ thy, it is the surface op. that measures the Arf inv. of the dynamical spin field.

Other components:

Since $\Omega\mathcal{B}$ is symmetric, so is $\Sigma\Omega\mathcal{B}$.
Need other components to make it nondeg. braided.

Defn: A string m is *magnetic* if 

$= -$ 

.

E.g.: in $\mathbb{Z}_2$ gauge thy, an 't Hooft surface.

It turns out that in all cases, $\pi_0 \mathcal{B} \simeq \mathbb{Z}_2 = \left\{ \begin{matrix} \text{id} \\ \text{comp} \end{matrix}, \begin{matrix} \text{mag} \\ \text{comp.} \end{matrix} \right\}$.

Furthermore, it turns out that $\exists$ an invertible magnetic string. I don't have a nice short argument for why. ([JF-Yu '20] notwithstanding.)

in bosonic case
m inv.
$m \cdot c$ not inv.

Note: $\pi_1 \mathrm{Aut}_{br} \mathrm{Rep}(\mathbb{Z}_2) = \pi_1 \mathrm{Aut}_{br} \mathrm{SVec} = $ " $\mathbb{Z}_2^{\varepsilon}$ " $= \{1, (-1)^{\varepsilon}\}$.

(and all other π_* vanish) i.e. $(-1)^{\varepsilon}$ is a 1-form symm.

We are looking for "extensions" (of braided 2-cats)

$$\Sigma \mathrm{Rep}(\mathbb{Z}_2) \cdot \mathbb{Z}_2^{\varepsilon} \quad , \quad \Sigma \mathrm{SVec} \cdot \mathbb{Z}_2^{\varepsilon}.$$

$$\cup\!| \qquad\qquad\qquad\qquad \cup\!|$$

$$(\Sigma \mathrm{Rep}(\mathbb{Z}_2))^{\times} \cdot \mathbb{Z}_2^{\varepsilon} \qquad (\Sigma \mathrm{SVec})^{\times} \cdot \mathbb{Z}_2^{\varepsilon}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\mathbb{C}^{\times}[2] \times \mathbb{Z}_2[1] \qquad \mathbb{C}^{\times}[2] . \mathbb{Z}_2[1] . \mathbb{Z}_2$$

$$\| \qquad\qquad\qquad \| \qquad \|$$

$$\{1, e\} \qquad \{1, e\} \quad \{1, c\}$$

E_2 spaces $\Leftrightarrow$ homotopy 4-types in deg. π_2, π_3, π_4.

These are classified by some twisted generalized cohomology.

Easy spectral sequence implies:

is unique (split), but there are 2 of these (split + non split).

Automorphism gps:

$\pi_0 \operatorname{Aut} \mathcal{R} = \mathbb{Z}_2$.

(id. functor. Change sign br. $\otimes$ data.)

$\pi_0 \operatorname{Aut} \mathcal{S} = \pi_0 \operatorname{Aut} \mathcal{T} = \mathbb{Z}_{16}$.

Odd elts switch

$m \Leftrightarrow mc$!

no canonical choice of magnetic string!

"Noether thm" for higher-form syms:

$$(*) \quad \operatorname{Aut} \mathcal{B} = \mathcal{A}^\times / \underbrace{Z(\mathcal{A})^\times}$$

$\leftarrow$ "scalars"

nondegenerate braiding $\Updownarrow$ $Z(\mathcal{A}) = \Sigma^2 \mathrm{Vec}$.

where $\mathcal{A} = \Sigma \mathcal{B}$ is the connected fusion 3-cat

obj = 2+1D "membranes" in the top. order.

$(*) \Rightarrow \pi_2 \operatorname{Aut} \mathcal{R} = \pi_2 \operatorname{Aut} \mathcal{S} = \pi_2 \operatorname{Aut} \mathcal{T} = \mathbb{Z}_2 = \{1, e\}$.

$\pi_1 \operatorname{Aut} \mathcal{R} = \mathbb{Z}_2 = \{1, m\}$. $\quad \pi_1 \operatorname{Aut} \mathcal{S} = \pi_1 \operatorname{Aut} \mathcal{T} = \{1, m, c, mc\}$.

Algebraic difference between $\mathfrak{Z}$ and $\mathcal{T}$:

Choose either magnetic string to call "m".

Choose either iso $m^2 \simeq \mathbb{1}$

e is a fermion

⇑ torsor for $\pi_0 \mathrm{Aut}(\mathbb{1}) = \{1, e\}$.

In $\mathbb{R}$ case, different choices here lead to dif. anomalies!

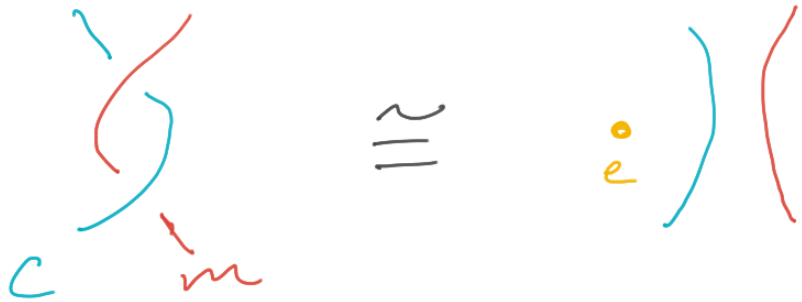
These choices are all permuted by the $\pi_0 \mathrm{Aut} = \mathbb{Z}_{16}$ action!

These choices $\rightsquigarrow$ action of $\mathbb{Z}_2[1]$ on $\mathfrak{Z}, \mathcal{T}$

↑ i.e. 1-form action.

In $\mathfrak{Z}$, it is nonanomalous.

In $\mathcal{T}$, it has nontrivial 't Hooft anomaly in $H^5(K(\mathbb{Z}_2, 2); \mathbb{C}^\times) \cong \mathbb{Z}_2$.



<u>Thm:</u> If

vertex ops = $\mathbb{C}$

"only one local vacuum"

then all

{particles, surfaces}

is nondeg. braided.

$Z(\mathcal{F}) = \text{triv.}$ | $\mathcal{F}$ = fusion (braided) 2-cat = 2-cat of states on S^0

$\rightsquigarrow$ "skein thy" for surfaces $\subseteq \mathbb{R}^3$ (4)

~~4~~ 5 & "bulk" thy use $[\mathcal{F}] \in \mathrm{Mor}_{\cancel{1}\,2}(2\mathrm{Cat})$ mor?

~~3~~ 4 & "boundry" uses $\mathcal{F}$.

If $\mathcal{A}$ fusion n-cat ($n = 3$)

w/ $\Omega^n \mathcal{A} = \mathbb{C}$

and $Z(\mathcal{A}) = \text{triv.}$

then $\mathcal{A} = \Sigma \Omega \mathcal{A}$. Thm!