McNamara – Wang Reconstruction

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SCGCS PI seminar

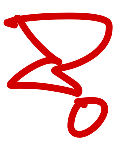
Theorem (McNamara–Wang, McN–JF–Reutter)

Let $\mathcal{B}$ be a rigid symmetric monoidal firm category with dagger + UDF. Any reflection-positive partition function

$$Z : \Omega\mathcal{B} \to \mathbb{C} \;\; (= \mathrm{End}(1))$$

extends uniquely up to non-unique iso to a unitary functor $\mathcal{B} \to \underbrace{\mathrm{sHilb}}_{\cap\!\!\mid\; \mathrm{sBan}}$.

e.g. $\mathcal{B} = \mathrm{Bord}^{\mathrm{Eucl.}}_{\langle n, n-1\rangle}$.



$\mathrm{Bord}^{\mathrm{Riem}}$ is not yet a dagger cat.

Quillen: If A ($= C_0(\text{some LCH space } X)$) is nonunital alg

$$\mathrm{Mod}_{\mathrm{nonunital}}(A) = \mathrm{Mod}_{\mathrm{unital}}(\underbrace{A \oplus \mathrm{unit}}_{C(X_+)})$$

$$\cup\!\!\mid$$

$$M$$

$$\begin{array}{ccc} A \otimes M & \longrightarrow & M \\ & \searrow \quad \nearrow \sim & \\ & A \otimes_A M & \end{array}$$

Defn: M is firm if $\sim$

E.g:

If A was unital, then

firm mods $\equiv$ unital modules

i.e. unit acts as identity.

Defn: A is firm if it is firm as a module over itself.

Categories $\equiv$ algebras with many objects.

presheaves $\equiv$ modules

"firm presheaf" = KS call "continuous"

C a nonunital cat, f some presheaf

$$\underbrace{\text{coend } \int^{y \in C} C(x,y) \times f(y)}_{\text{"} C \otimes_C f \text{"}} \to f(x)$$

monoidal cat

C is **firm** if all representable presheaves are firm.

E.g. Basically any geometric bordism category

Unpacks to: $\forall$

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr] & & y \\ & z \arrow[ur] & \end{tikzcd}$$

$\exists$

and

$$\begin{tikzcd} x \arrow[rr] \arrow[d] \arrow[drr] & & y \\ z \arrow[urr] & & w \arrow[u] \end{tikzcd}$$

$\exists$ ....

A is firm if

$A \otimes_A A \to A$ is iso.

Everything makes sense enriched in any presentable sym $\otimes$ $\mathcal{V}$.

Thm: (Ramzi) If C is a firm $\mathcal{V}$-cat then $\mathrm{Psh}^{\mathrm{firm}}(C) \in \mathrm{Mod}_{\mathcal{V}}^{\mathrm{Pr}}$ is 1-dualizable in $\mathrm{Mod}_{\mathcal{V}}^{\mathrm{Pr}}$ and moreover every 1-dualizable object in $\mathrm{Mod}_{\mathcal{V}}^{\mathrm{Pr}}$ is $\mathrm{Psh}^{\mathrm{firm}}(C)$ on some firm C.

In fact: Given $\mathcal{M} \in (\mathrm{Mod}_{\mathcal{V}}^{\mathrm{Pr}})^{1d}$ pick a dense small subcat $D \subset \mathcal{M}$.

$$\mathcal{M}(d,-) : \mathcal{M} \to \mathcal{V}$$

define $\mathcal{M}^{at}(d,-)$ to be the "atomic"

$\mathcal{M}^{at}(d,-) : \mathcal{M} \to \mathcal{V}$ defined

universal colimit-preserving functor

w/ $\mathcal{M}^{at}(d,-) \Rightarrow \mathcal{M}(d,-)$.

Given $D \subset \mathcal{M}$,

$$C = \left\{ \begin{array}{l} \text{objects}(D) \\ \hom(d,d') = \mathcal{M}^{at}(d,d') \end{array} \right\}$$

is firm and $\mathcal{M} = P_{Sh}^{firm}(C)$.

E.g.: If $\mathcal{M} = \mathcal{V}$, then

$$\mathcal{M}^{at}(d,d') = d^\vee \otimes d'$$

E.g. If $\mathcal{V} = \mathrm{Ban}$, then

$\mathcal{M}^{at}$ = kernels of nuclear operators.

E.g.: Hilb $\subset$ Ban is dense,

nuclear ops between Hilbert spaces $\equiv$ trace-class operators.

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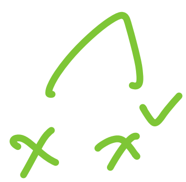
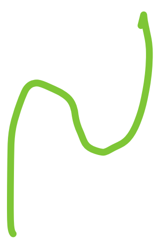
usual defn of "rigid $\otimes$"

is $\forall x \ \exists x^\vee$ and



and

s.t.



$= \ |\ \mathrm{id}\ x.$

w/o identities, use Gaitsgory-rigidity

If C is a monoidal (fin) cat,

$$\begin{tikzcd}
C \times C \times C \arrow[r, "C \times \otimes"] \arrow[d, "\otimes \times C"'] \arrow[dr, phantom, "\alpha \sim \Rightarrow"] & C \times C \arrow[d, "\otimes"] \\
C \times C \arrow[r, "\otimes"'] & C
\end{tikzcd}$$

Take fin presheaves to get into presentable world.

$(C, \otimes)$ is Gaitsgory-rigid if this square becomes a Beck–Chev square.

In general, if $C, \otimes$ is rigid then has a (unique) $C \xrightarrow{(-)^\vee} C^{op}$.

If C is also t, then UDF is a compat between $(-)^\vee$ and T.

Given $\mathcal{B}$ some rigid sym $\otimes$ fin cat, linearize to $\mathbb{C}\mathcal{B}$.

Suppose $\mathcal{B}$ is $\dagger$ w/ UDF.

Then so is $\mathbb{C}\mathcal{B}$.

Get $\langle f, g\rangle = \mathrm{Tr}_x(f^\dagger g) \in \Omega\mathbb{C}\mathcal{B}$.

uses UDF

on $\hom(x,y)$

Call an element of $\Omega\mathbb{C}\mathcal{B}$ <u>positive</u> if it is $\langle f,f\rangle$ "=" $|f|^2$ for some morphism $f \in \mathbb{C}\mathcal{B}$.

A $*$-homomorphism $Z : \mathcal{B} \to \mathbb{C}$ is <u>reflection</u> _<u>positive</u> if it takes positives to positives.

Steps of proof:

Given (B, Z) "positive tracial state on B"

① Run a GNS construction.

$\rightsquigarrow$ $\forall\, b \in B$, a rep'n "universal construction"

$$\underbrace{\mathcal{H}_b}_{\text{natural in } b.} : B \to \mathrm{Hilb}$$

Get $B \to \mathrm{Fun}^{\mathrm{fin}}(B, \cancel{\mathrm{Hilb}}\ \mathrm{Ban})$.

② Extract a fin/m subcat $\mathcal{H}_B$

$\mathrm{obj} = \mathrm{ob}(B)$

$\mathrm{hom}(b, b') = \mathrm{hom}^{\mathrm{cat}}(\mathcal{H}_b, \mathcal{H}_{b'})$.

some Banach cat.

$B \to \mathcal{H}_B$.

(3) Colafranceschi – Dong – Marolf – Wang:

This fin $\mathcal{H}_B$ is Morita equiv to a $\underbrace{\text{unital cat}}_{\text{essentially } \mathrm{Kar}(\mathcal{H}_B)}$.

Explicitly:

$$\mathrm{Psh}^{\mathrm{fin}}_{\mathrm{Ban}}(\mathcal{H}_B)$$

is atomically generated

the only vN alg you get are $\oplus$ Type I.

(4) Recognize:

$\mathrm{Kar}(\mathcal{H}_B)$ is still rigid sym $\otimes$ + w/ UDF but now unital (and C^*).

Then use Doplicher–Roberts reconstruction. $\square$

Thm (Doplicher-Roberts)

If C is a (unital) rigid sym $\otimes$ C^*-category then

$$\pi_0 \hom_{C^*\text{sym}\otimes}(C, \text{sHilb}^{f.d.}) \xrightarrow{\Omega} \hom_{C^*\text{cog}}(\Omega C, \mathbb{C})$$

is iso.