Homotopy Quantum Groups

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Based on joint work in progress w/ David Reutter and conversations throughout this collaboration.

These slides: http://categorified.net/SCGCS-hypergroups.pdf

A Change of Perspective

not necessarily extended!
not necessarily compact!

We normally picture TQFTs as an assignment

manifolds $\rightsquigarrow$ algebraic data

w/ a cutting-and-gluing rule. This picture is incomplete.

Compare:
- stacks: {affine schemes} $\longrightarrow$ {groupoids}
- cohomology theories: {spaces} $\longrightarrow$ {abelian groups}.

In these examples, it is useful to picture instead a "representing object" that is an object of geometry / topology.

Similarly, TQFTs should be pictured as quantum topological spaces.

They are things that you can do algebraic topology to.

Very first step: Define and analyze π_*(a TQFT).

π_0(a TQFT)

A classical space X ⇝ a quantum space: the $n+1$D σ-model w/ target X.

This "truncates" X, throwing away information of high homotopical degree.

Best case: X is an $(n-1)$-type. (3D DW thy knows G. 2D DW thy only knows $\# \mathrm{Irr}(G)$.)

i.e. $\pi_{\geq n} X = *$.

Exercise: In this best case, what is, say, Hilbert space of S^n?

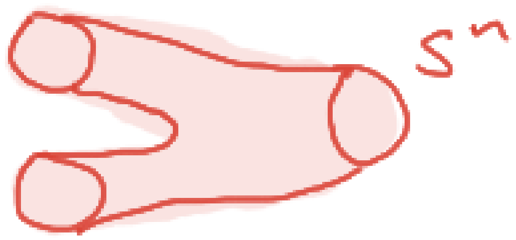
Answer: $\mathbb{C}(\mathrm{maps}(S^n, X)) = \mathbb{C}(\pi_0 \mathrm{maps}(S^n, X)) = \mathbb{C}(\pi_0 X)$.

since X is an $(n-1)$-type.

This vector space knows $|\pi_0 X|$!

Moreover, in this example, we have a com. alg str (product of functions).

We can recover $\pi_0 X$ as its spectrum!



But actually for any $n+1$D TQFT Q, $Q(S^n)$ is a com. alg. via pair of pants.

These algebra structures agree.

π_0(a TQFT)

(Sweedler? Kontsevich?)

Remark: $\pi_0 X = \operatorname{Spec} \mathbb{C}(X)$ is slightly false if $|X| = \infty$.
This can be fixed by working instead w/ distributions and coalgebras.
Even in the finite case, it will be convenient later to use coalgebras.

Definition: Given an $n+1$D TQFT Q,

$$\pi_0 Q := \operatorname{cospec}\left(Q(S^n),\ \text{copants}\right)$$

(Grothendieck)

Remark: If a coalg A is cosemisimple, then $\operatorname{cospec}(A)$ is its set of grouplike elements. In general, "(co)spec" has no mathematical content. It simply means "think spacially".

π_k (a quiche)

The (≥ 1)-homotopy gps of a space X are not well-defined: they depend on a choice of basepoint $x : pt \to X$.
Such a choice selects a Dirichlet boundary condition for the σ-model, in other words it extends the σ-model to an open-closed TQFT.

Definition (Freed–Moore–Teleman): An nD quiche is an $n{+}1$D TQFT Q equipped with an nD boundary condition $\mathcal{q}$.
These are the things that act on nD QFTs.

In the space case, $\Omega^k(X, x) := \mathrm{maps}(D^k \to X,\ \partial D^k \to x)$, and $\pi_k := \pi_0 \Omega^k$.

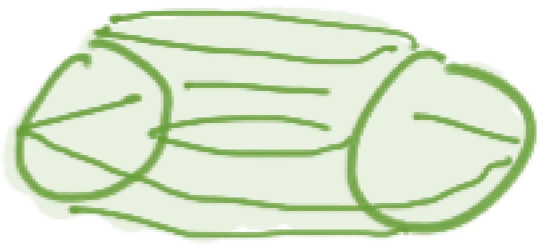
Defn: $$\Omega^k(Q, \mathcal{q}) : M \mapsto (Q, \mathcal{q})(D^k \times M)$$

$$\pi_k(Q, \mathcal{q}) := \pi_0 \Omega^k(Q, \mathcal{q}) = (Q, \mathcal{q})(D^k \times S^{n-k})$$

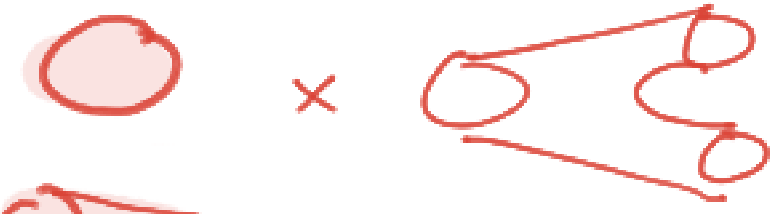
compactification on D^k to produce an $n{-}k{+}1$ D TQFT.

Note: If X is "best", i.e. an $(n-1)$-type, then $\Omega^k(X, x)$ is an $(n-k-1)$-type, also "best".

π_k (a quiche)

$$\pi_k(Q, q) = (Q, q)(D^k \times S^{n-k}) =$$ 

solid torus,
Q in bulk,
q on boundary.

This is:
- a coalgebra via $D^k \times$ copants. 
- an algebra via chaps $\times S^{n-k}$. 

In the case of a pointed space (X, x):
- the coalgebra structure encodes $\pi_k(X, x)$ as a set.
- the algebra structure encodes the group law.

Equivalently: π_k(σ-model for X, Dirichlet b.c. for x) is a Hopf algebra, specifically it is the group algebra $\mathbb{C}[\pi_k(X,x)]$.

Is this a universal phenomenon? No.

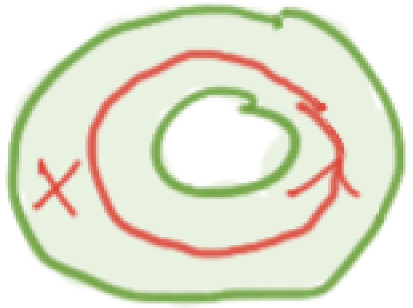
Fusion Rings

Let $\mathcal{C}$ be a spherical fusion category, and (Q, q) its Barrett-Westbury-Levin-Wen-Turaev-Viro TQFT, with its standard b.c.

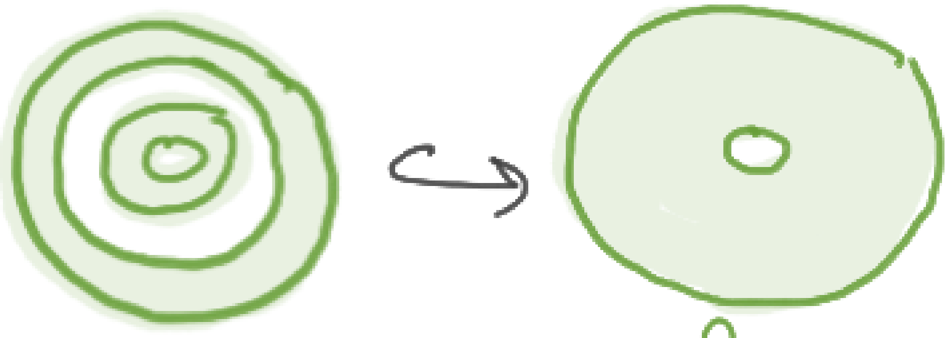
spherical in this telling, but this can be dropped — it still works!

Exercise: As an algebra, $\pi_1(Q, q) = K_0(\mathcal{C}) \otimes \mathbb{C}$ is the fusion ring of $\mathcal{C}$. Its standard basis is the classes $[X]$ for simple X. Show that the coalgebra structure is

$$\Delta([X]) = \frac{1}{\dim X}[X] \otimes [X], \quad \varepsilon([X]) = \dim X.$$



$D^1 \times S^1$



mult is the bordism of stacking annuli

In other words, the distinguished basis is

$$\mathrm{Cospec}(\text{fusion ring}) = \left\{ \frac{[X]}{\dim X}, \ X \text{ simple} \right\}, \quad \text{not } \{[X]\}.$$

This is Hopf only if the basis is closed under $\times$, which it typically isn't. i.e. Δ is not an alg. hom. (But ε is.)

For fusion n-cats, get simples up to condensation

Neumann Boundary

A thing you can do with quantizations of spaces that you cannot do classically is to take superpositions of pointings: point by a map $Y \to X$ w/ $Y \neq *$. E.g.: $X \xrightarrow{id} X$ gives the Neumann b.c.

Exercise: If $\pi_0 X = pt$,* then $\Pi_K(X, \text{Neumann})$ is the $\pi_1 X$-invariant subalgebra of $\mathbb{C}[\pi_K X]$. The distinguished basis is the average over an orbit.

Technical condition: $\pi_1 X \circlearrowright \pi_K X$ has finite orbits.

In other words, $\Pi_K(X, \text{Neumann}) = \pi_K(X)/\pi_1(X)$.

This is not available in classical homotopy theory, because $\pi_K(X)/\pi_1(X)$ is rarely a group. But is allowed quantumly.

* In general, get $\bigoplus_{\pi_0 X} \Pi_K(X, \text{inclusion of connected component})$.

Hypergroups

the one selected by cosemisimple Δ.

In all of these examples, $1 \in$ basis. Moreover, $\forall a, b \in$ basis,

$$\sum_{c \in \text{basis}} m_{ab}^c = 1$$

matrix coef. of mult.

Interpretation: m_{ab}^c is not the number of $a \times b \to c$ fusion channels, but the probability* of an $a \times b \to c$ fusion. * Not nec. in $\mathbb{R}_{\geq 0}$.

Moreover, there is an antiautomorphism $a \mapsto a^*$ called antipode s.t. $\forall a \in$ basis, a^* is the unique basis vector w/ $m_{aa^*}^1 \neq 0$.

Definition: These are the axioms for a hypergroup.

Theorem*: If (Q, q) is an nD quiche with $\pi_n Q = \mathbb{C}$, then $\pi_k(Q, q)$ is a hypergroup, $k = 1, \ldots, n-1$.

* Compact w/ stable tangential str. General case in progress.

For a purely algebraic defn that works when Δ is not cosemisimple, see Delvaux – Van Daele.

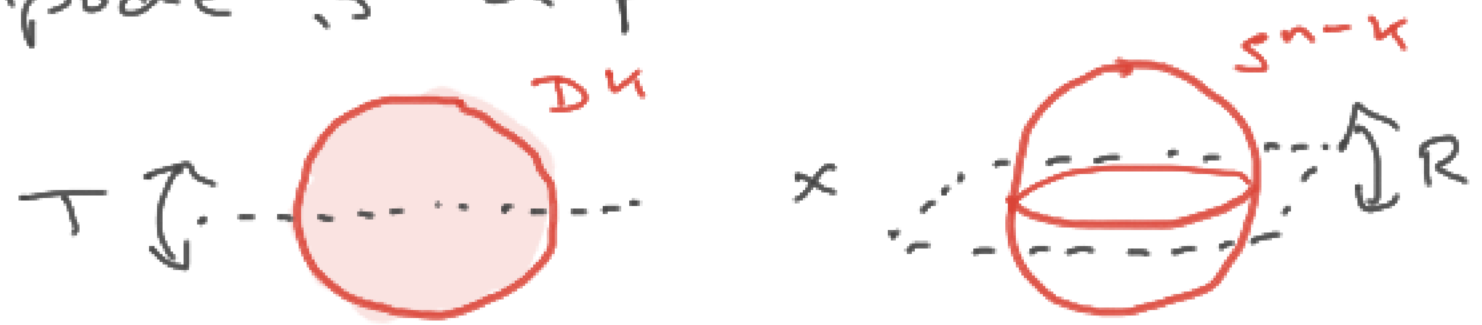
Antipode Map

* Thm in compact case.

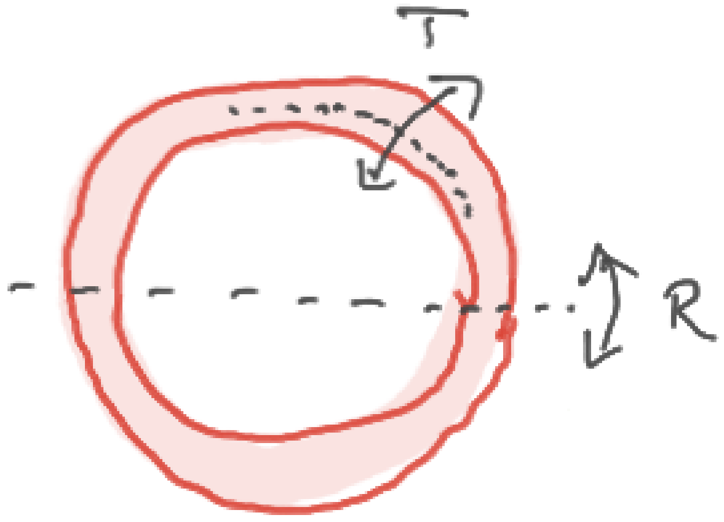
Theorem*: If $(Q,\mathfrak{g})$ is an nD quiche with $\pi_n Q = \mathbb{C}$, then $\pi_k(Q,\mathfrak{g})$ is a hypergroup, $k=1,\dots,n-1$.

needed even for non-antipode axioms. Otherwise, π_k is a hypergpoid rather than a hypergp.

Antipode is a product of two reflections (plus rotate the framings).



Interpretation: compactify to $\Omega^k(Q,\mathfrak{g})$. The D^k directions become "temporal", so R is a "space" reflection by T is a "time" reflection. Their composition, the antipode, conjugates charge.



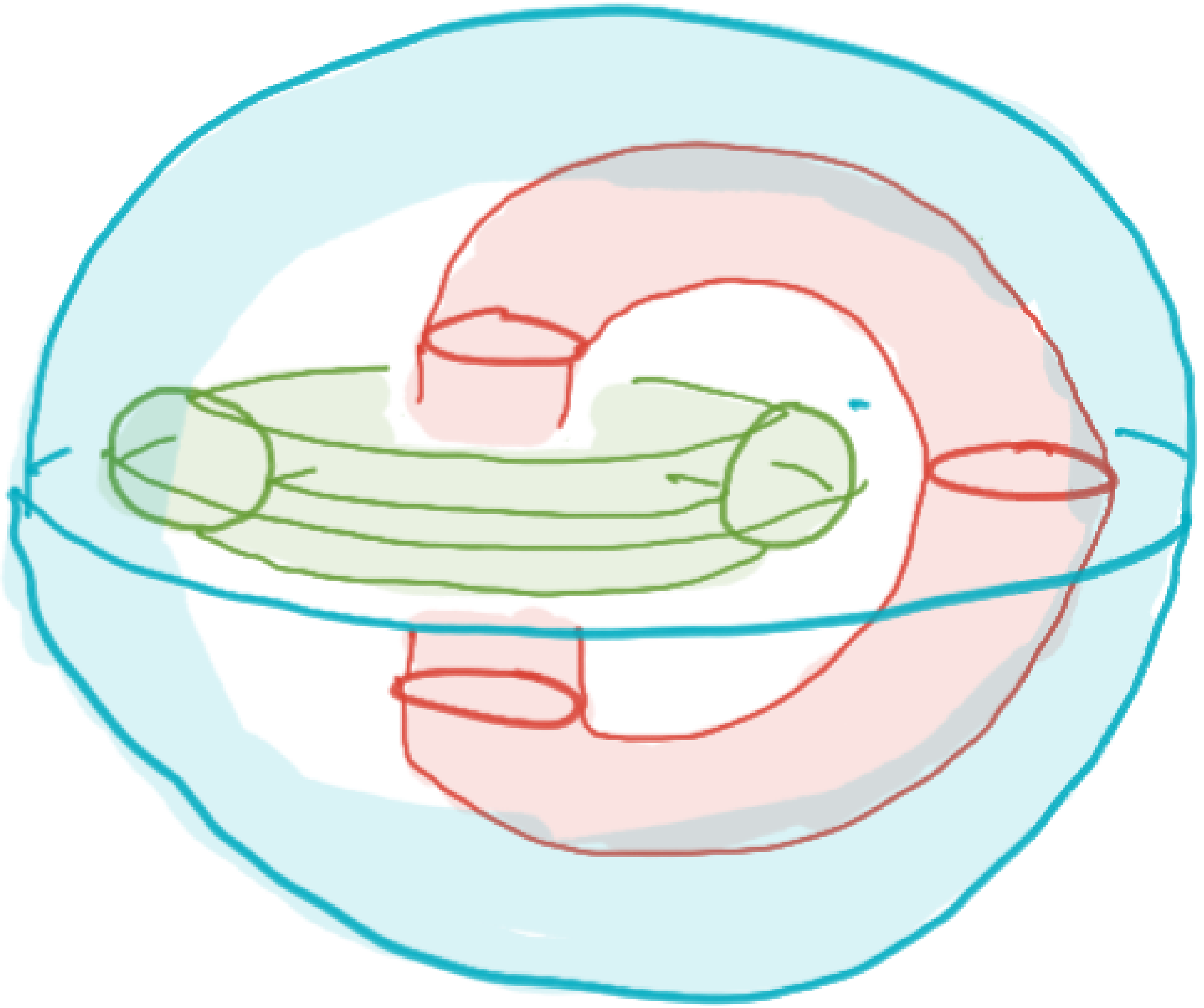
$D^1 \times S^1$

this is a CRT Theorem for TQFTs.

No unitarity, extendedness, ...

S-matrices & Poincaré Duality

Consider the Hopf link $D^k \times S^{n-k} \hookrightarrow D^{n-k+1} \times S^{k-1} \hookleftarrow D^n$.



The bordism of this embedding defines an S-matrix pairing

$$S_k : \pi_k(Q,g) \times \pi_{n-k+1}(Q,g) \to (Q,g)(D^k) \underset{\text{in hypergroup case.}}{=} \mathbb{C}$$

Theorem*: S_k is nondeg $\forall k$ iff bulk Q is invertible.

*If a semisimplicity assumption, e.g. if once extended. General case in progress.

Verlinde formula: If S_k is nondeg, then the fusion probs. for π_k are

$$m_{ab}^{c} = \sum_{x \in \pi_{n-k}} S_{a,x} \cdot S_{b,x} \cdot (S^{-1})_{x,c}$$

Pf: Immediate: S is a hypgp pairing.

Quantum Puppe Sequence?

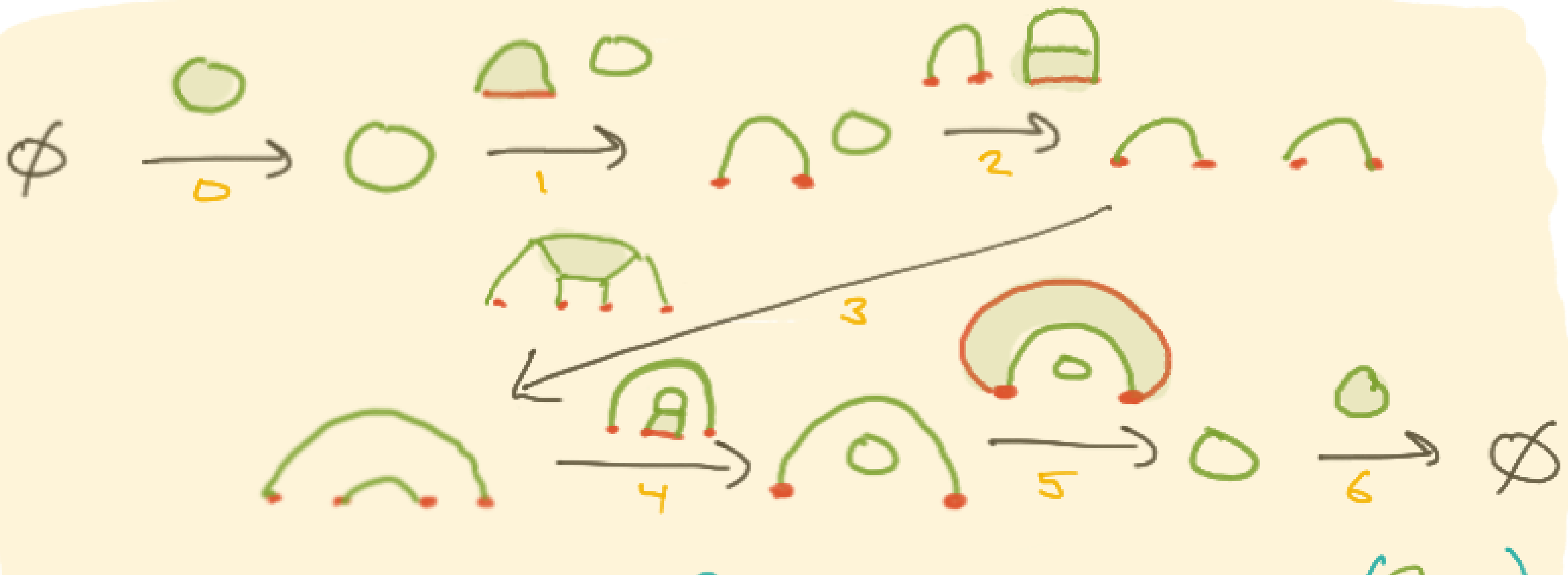
Main ingredient in our S-matrix theorem:
for any (semisimple, so far) quiche $(Q, \mathfrak{q})$,
and more generally for any quiche square,
there is a LES relating its various π_k's.

It seems to be a version of the Puppe sequence.

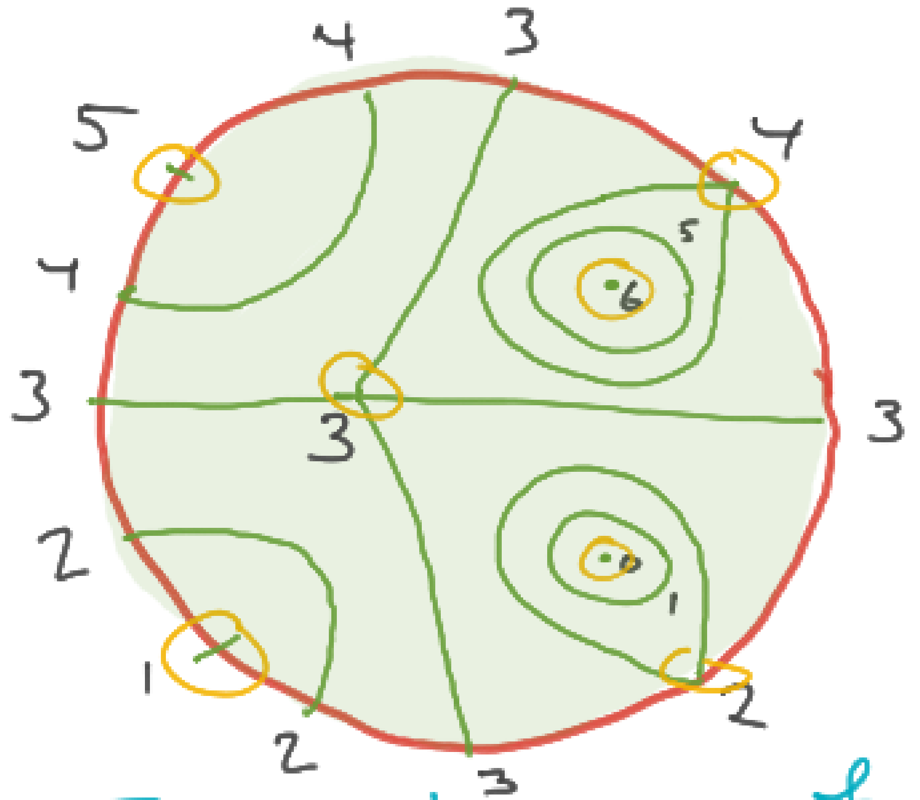
The full story is work in progress.

$$\begin{tikzcd} \text{defect} \arrow[r] \arrow[d] & \text{b.c.} \arrow[d] \\ \text{b.c.} \arrow[r] & \text{TFT} \end{tikzcd}$$

a quiche square.



Our sequence for a 1D quiche $(Q, \mathfrak{q})$



Topographic map of the composite jigsaw puzzle

Future Questions

- Fix Q but vary g. The hypergroups $\pi_k(Q,g)$ change up to iso. Are they somehow "the same" in a weaker sense?
- The S-matrix is a specific Whitehead bracket. What about the other ones?
- What is the Postnikov extension data that links the hypergroups together? Is there a hypergroup cohomology?
- We can define homotopy hypergroups without full compactness, extendedness, etc. Explore how general you can go.
- Are there homotopy hypergroups for dynamical QFTs?
- What else can you learn from the quantum Puppe sequence?

THANKS!

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Post script: (Non-)Invertibility

Already in "semiclassical" examples $Y \to X$, π_k can be truly hyper. This is a type of non-invertibility at the level of homotopy, i.e. up to condensation. But it is caused by a sort of "non-simply-connectedness":

Theorem (follows from JF-Yu): For quiches arising from fusion n-categories, $n \geq 3$, all homotopy hypergps are $\pi_k = \mathbb{C}[\text{finite abelian } A_k] / \text{finite gp } G$. Moreover, G does not depend on k, and can be canonically ungauged to produce a quiche with all π_k honest groups.

The truly non-invertible/quantum nature is not in π_0 (fusion rules) but in the extension data, which definitely does go beyond classical homotopy theory.