Crash course in Tannaka-Krein theory

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Abstract

Tannaka-Krein theory asks two main questions: (Reconstruction) What about an algebraic object can you determine based on knowledge about its representation theory? (Recognition) Which alleged "representation theories" actually arise as the representation theories of algebraic objects? In this talk I'll mention some answers to the second question, but I'll focus more on the first. The punchline: essentially everything, provided you remember the underlying spaces of your representations — there is an almost perfect dictionary between algebraic structures and categorical structures. My goal is to explain the results in as elementary and pared-down a way as possible, so the talk will be more or less reverse-chronological. The only prerequisite is some brief acquaintance with the following two-categories: (Category, Functor, Natural Transformation) and (Algebra, Bimodule, Intertwiner). The main Tannaka-Krein story that I will present is "twentieth century" and by now well known, but time permitting I will also mention some joint work in progress with Alex Chirvasitu.

0 Warm-up: Yoneda's lemma

Groups

Let G be a group (or just a monoid). Then:

$$ {}^{G}\text{SET} \stackrel{\text{def}}{=} \{\text{“permutation representations” of } G\} $$

Think of G as the category "BG" or "$1/G$", which I'll just write as G, which has one object $\star$ and $\mathrm{Hom}(\star,\star) = G$. Then:

$$ {}^{G}\text{SET} = \text{Functors}(G \to \textsc{set}) $$

Yoneda's lemma: We have a full faithful embedding $G^{\mathrm{op}} \hookrightarrow {}^{G}\text{SET}$. It sends $\star \mapsto \mathrm{Hom}(\star, -) = {}_{G}G = G$ with its left multiplication action, and $g \mapsto$ left multiplication by g. The content of Yoneda's lemma is that $\mathrm{End}({}_{G}G) = G^{\mathrm{op}} = G$ acting on the right.

(**Yoneda's lemma**)^2**:** We have a full faithful embedding $G \hookrightarrow \text{Functors}({}^{G}\text{SET} \to \textsc{set})$ given by $\star \mapsto \mathrm{Hom}_{{}^{G}\text{SET}}({}_{G}G, -) = \text{Forget} : {}^{G}\text{SET} \to \textsc{set}$. Rewriting this, the group G is precisely the natural endomorphisms of the functor Forget.

Rings

We can replace SET by some other category. For example, abelian groups or vector spaces. Let's fix a commutative ring K and set $\mathcal{K} = K\text{-MOD}$ (with $\otimes = \otimes_K$). Then a *$\mathcal{K}$-enriched (= K-linear) category* is a category where all hom sets come equipped with the structure of objects in $\mathcal{K}$, i.e. K-modules, and composition is K-bilinear.

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ assigns an object of $\mathcal{D}$ to each object of $\mathcal{C}$, and for each pair $X, Y \in \mathcal{C}$ the functor gives a function $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$. (And there are compatibility conditions.) A functor is *$\mathcal{K}$-enriched* or *K-linear* if this function is K-linear (an arrow in $\mathcal{K}$).

Example: if A is a K-algebra, then there is a $\mathcal{K}$-enriched category "$1/A = \mathrm{B}A$", but I'll just call it A, with one object $\star$ and $\mathrm{End}(\star) = A$. We set ${}^A\mathcal{K} = A\text{-MOD} = \mathrm{Functors}(A \to \mathcal{K})$. (**Exercise:** the usual notion of A-module is the same as the notion of K-linear functor.)

Then the same argument gives $A = \mathrm{End}(\mathrm{Forget} : {}^A\mathcal{K} \to \mathcal{K})$. **Exercise:** Without using Yoneda, prove this directly.

1 Coalgebras and profinite algebras

Pro f.d. algebras

Now I will force K to be a field. If you want to work over a ring ($\mathbb{Z}$), then "finite dimensional" should be replaced with "finitely generated projective" or "dualizable" or "compact projective" or ....

What if we take our category $\mathcal{K}$ to be not all K-modules (vector spaces), but just the finite-dimensional ones? Then we still have a Yoneda lemma, so we can still recover finite-dimensional algebras from their finite-dimensional modules. But some algebras have very few finite-dimensional modules, e.g. $K = \mathbb{C}$ and $A = \mathbb{C}(x)$. So if A is some K-algebra that's not finite dimensional, and

$$E \overset{\text{def}}{=} \mathrm{End}(\mathrm{Forget} : {}^A\text{F.D.VECT} \to \text{F.D.VECT})$$

then we often don't have $E = A$. We always have a map $A \to E$ — A acts on every module. This map is an iso iff A is *pro finite dimensional*, i.e. it is the projective limit of its finite dimensional quotients. (E has a topology on it, which is the coarsest topology such that $E \to \mathrm{End}(\mathrm{Forget}(X))$ is continuous for every finite-dimensional X; this is the *profinite topology*.)

This map $A \to E$ is the *profinitization of* A. E is the universal profinite algebra with a map from A.

Coalgebras

We could also take $\mathcal{K} = (\text{K-VECT})^{\mathrm{op}}$. Then an algebra in $\text{VECT}^{\mathrm{op}}$ is a *coalgebra* in $\mathcal{K}$, i.e. a vector space A with a map $A \to A \otimes A$, satisfying some conditions. An *A-comodule* is an A-module in $\text{VECT}^{\mathrm{op}}$, i.e. a vector space X with a map $X \to A \otimes X$ satisfying some conditions, and ${}^A(\text{VECT}^{\mathrm{op}}) = (A\text{-COMOD})^{\mathrm{op}}$.

Let's spell out what "natural endomorphisms of $\mathrm{Forget} : {}^A(\text{VECT}^{\mathrm{op}}) \to \text{VECT}^{\mathrm{op}}$" means. By analogy, just think about algebras. Then End(Forget) is the equalizer of:

$$\prod_{X \in A\text{-MOD}} \mathrm{End}(\mathrm{Forget}\, X) \overset{\circ f}{\underset{f\circ}{\rightrightarrows}} \prod_{\substack{X,Y \in A\text{-MOD} \\ f : X \to Y}} \mathrm{Hom}(\mathrm{Forget}\, X, \mathrm{Forget}\, Y)$$

So "op"ing everything turns products into coproducts and equalizers into coequalizers.

Lemma: Every coalgebra is a union of its finite dimensional sub coalgebras. Every comodule is a union of its finite-dimensional sub comodules. The slogan: Coalgebras = ind-f.d. coalgebras.

Corrolaries:

1. To know a coalgebra, it suffices to know its finite dimensional comodules.

   But we have an equivalence $\text{F.D.VECT}^{\text{op}} \xrightarrow{*} \text{F.D.VECT}$, and so by the above observations, any coalgebra A is the coequalizer of the diagram

   $$\coprod_{X \in A\text{-F.D.COMOD}} \operatorname{End}(\operatorname{Forget} X)^* \underset{f\circ}{\overset{\circ f}{\rightrightarrows}} \coprod_{\substack{X,Y \in A\text{-F.D.COMOD} \\ f : X \to Y}} \operatorname{Hom}(\operatorname{Forget} X, \operatorname{Forget} Y)^*$$

   But $\operatorname{End}(X)^* = \operatorname{End}(X)$ when X is finite-dimensional, and $\operatorname{Hom}(X, Y)^* = \operatorname{Hom}(Y, X)$ when both are finite-dimensional.

   Coequalizers of diagrams like the one above determine the notion of *natural cotransformation* of functors (it makes sense provided the essential images of the functors consist entirely of dualizable objects).

2. The category of coalgebras is opposite to the category of profinite algebras. **Proof:** $\text{F.D.COALG}^{\text{op}} \overset{*}{\leftrightarrow} \text{F.D.ALG}$.

3. The map {algebra → its finite dimensional modules → coalgebra → algebra} agrees with the "profinite completion" of an algebra. When the algebra is a group algebra of a group, we'll see that the reconstructed coalgebra is the coalgebra of functions on an affine algebraic group. The corresponding affine group scheme is the *algebraization* of the group.

Exercise: What is a coalgebra/comodule in SET? Is every coalgebra/comodule determined by its finite subs?

2 Higher structures

Morphisms

A *Tannakian category over* $\mathcal{K}$ is a $\mathcal{K}$-enriched category $\mathcal{A}$ with a $\mathcal{K}$-enriched functor $F : \mathcal{M} \to \mathcal{K}$. The functor F is called the *fiber functor*. Warning: most texts require more adjectives.

Example: Any (pro) algebra A in $\mathcal{K}$ determines a Tannakian category $\mathcal{A} = A\text{-MOD}$, $F = \operatorname{Forget}$. We saw that $(\mathcal{A}, F)$ recovers A.

There are various types of functors of Tannakian categories. The one I will take:

- A 1-morphism $(\mathcal{A}, F) \to (\mathcal{B}, G)$ is a $\mathcal{K}$-enriched functor $E : \mathcal{A} \to \mathcal{B}$ and a natural isomorphism $\alpha : F \cong G \circ E$, i.e. it is a triangle.

- A 2-morphism is a cone on a bigon.

Now again I'll work over $\mathcal{K} = K\text{-MOD}$ for K a ring. Pick algebras A, B and $\mathcal{A} = A\text{-MOD}$ and $\mathcal{B} = B\text{-MOD}$, and take the forgetful maps as the fiber functors"

Exercise: Any 1-morphism is exact, cocontinuous, faithful, etc.

Corollary: To know $E : \mathcal{A} \to \mathcal{B}$, it suffices to know $E({}_A A)$.

But ${}_A A$ has a right A action, and so $E({}_A A)$ does too, i.e. $E({}_A A_A) = {}_A \square_B$ for some $\square$. But $\alpha : \operatorname{Forget}({}_B \square_A) \cong \operatorname{Forget}({}_A A_A) = A$. So $E({}_A A_A)$ is some module of the form ${}_B A_A$, and this data is the same as an algebra homomorphism $f : B \to A$. And $E = f^* =$ pull back along f.

What about the 2-morphisms? **Exercise:** take algebra homomorphisms $f, g : B \rightrightarrows A$. A 2-morphism between $f^*, g^* : A\text{-MOD} \rightrightarrows B\text{-MOD}$ is the same as an element $a \in A$ such that for all $b \in B$, $a \cdot f(b) = g(b) \cdot a$.

So we have given a full faithful injection of 2-categories:

$$\{\text{algebras, homomorphisms, conjugations}\} \hookrightarrow \{\text{Tannakian categories}\}$$

Tensor products

There is a symmetric tensor product $\otimes$ of algebras.

There is also a symmetric tensor product $\boxtimes$ of (enriched) categories, and hence of Tannakian categories, via:

$$(\mathcal{M}, F) \boxtimes (\mathcal{N}, G) = (\mathcal{M} \boxtimes \mathcal{N}, \mathcal{M} \boxtimes \mathcal{N} \stackrel{F \boxtimes G}{\to} \mathcal{K} \boxtimes \mathcal{K} \stackrel{\otimes}{\to} \mathcal{K})$$

You probably know that $(A\text{-MOD}) \boxtimes (B\text{-MOD}) = (A \otimes B)\text{-MOD}$ (a highly nontrivial theorem!). So this embedding is an embedding of symmetric monoidal 2-categories.

Remark: The essential image consists of those Tannakian categories $(\mathcal{M}, F)$ that are: cocomplete, and $F = \mathrm{Hom}(X, -)$ where X is a compact projective generator of $\mathcal{M}$. If you want, you can include these in the definition of "Tannakian category" and make an equivalence.

Warning: Most people don't work over VECT, but just over F.D.VECT. Then the essential image of PRO.F.D.ALG = COALG is: abelian, and F is exact and faithful.

3 Dictionary

Since we have an equivalence of categories, we get a perfect dictionary between algebraic structures and categorical / representation theoretic structures:

Tannakian category $\mathcal{A}$	algebra A
$\otimes : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ (which is iso to $\otimes_K$ upon forgetting)	homomorphism $A \to A \otimes A$
associator for $\otimes$ (some 2-iso satisfying a "pentagon equation")	invertible element of $A^{\otimes 3}$ intertwining the two homomorphisms $A \to A^{\otimes 3}$ satisfying some pentagon "quasicoassociative bialgebra"
associator is trivial upon forgetting	this element is 1 "bialgebra"
braiding	"quasitriangular structure"
symmetry	"triangular structure "
that is trivial upon forgetting	A is cocommutative, and the triangular structure is the canonical one
good theory of duals	antipode, so A is "Hopf"

Some explanation of the last line: If A is a (quasi)bialgebra, then there is an *inner hom*

$${}_A\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}({}_A A_A \otimes {}_A X \to {}_A Y)$$

where the hom is taken for the left A action (diagonal on the $A \otimes X$), and the A action on the inner hom is the right A action on ${}_A A_A$. **Exercise:** there is always a map $\mathrm{Forget}({}_A\underline{\mathrm{Hom}}(X, Y)) \to \mathrm{Hom}(\mathrm{Forget}\, X, \mathrm{Forget}\, Y)$. By "good theory of duals" we mean that this map is always an iso.

Break

4 Some algebraic geometry

What Deligne does

Of course, if you want to look at only finite-dimensional modules, you can read the same dictionary as above with "co"s everywhere. Then the point is that the word "commutative Hopf algebra" is the same as "affine algebraic group".

You may know from Hopf algebra theory that grouplike elements in the group ring KG are precisely the elements of the group G. But, upon recognizing KG as the natural endomorphisms of Forget : ${}^G\mathcal{K} \to \mathcal{K}$, where $\mathcal{K} = K\text{-MOD}$, a little unpacking shows that the grouplike elements are precisely the monoidal natural endomorphisms.

A very similar story is true for affine algebraic groups: the K-points of an affine algebraic group $A = \mathcal{O}(G)$ are precisely the monoidal natural transformations of Forget : ${}^G\text{F.D.VECT} \to \text{F.D.VECT}$. (It's surprising that all such endomorphisms are automorphisms, but follows from ${}^G\text{F.D.VECT}$ having a good theory of duals.)

If you really want a *group scheme*, you do the following: Let $(\mathcal{M}, F : \mathcal{M} \to \text{F.D.VECT})$ be a symmetric monoidal Tannakian category. Then for each commutative K-algebra R, consider $\mathcal{M} \overset{F}{\to} \text{VECT} \overset{\otimes R}{\to} R\text{-MOD}$. Since $\otimes R$ is a monoidal map (where the tensor structure on R-MOD is $\otimes_R$), this composition is monoidal. We define a group scheme by setting $G(R) = \{$the monoidal endomorphisms of this composition$\}$. So it is usual to define "Tannakian category" as something over F.D.VECT, with symmetric tensor structure and a good theory of duals.

An advertisement: what if you don't remember Forget?

I'd like to close with the following question, and an advertisement for some current joint work in progress. The question is: what can you recover from the category of representations if you don't keep track of which functor is Forget — i.e. if you just remember the (symmetric monoidal) category up to (symmetric monoidal) equivalence?

When the category is simply the representation theory of a ring, the answer is well known. The correct functors of cocomplete (compact-projective-generated) categories are the cocontinuous ones, and cocontinuous functors between module categories of rings exactly correspond to bimodules. In particular, an equivalence between A-MOD and B-MOD is a *Morita equivalence* — it is the same data as a pair of bimodule ${}_AM_B$, ${}_BN_A$ and isomorphisms ${}_AM_B \otimes_B {}_BN_A \cong {}_AA_A$ and ${}_BN_A \otimes_A {}_AM_B \cong {}_BB_B$. The classical example of a Morita equivalence is for any ring K, the rings $\text{Mat}(n, K)$ are all equivalent, as witnessed but the bimodules K^{nm}.

What about representations of groups? When K is an algebraically closed field, a theorem of Deligne's is that ${}^G\text{VECT}$ (or ${}^G\text{F.D.VECT}$) has a unique-up-to-isomorphism symmetric monoidal (and a few more adjectives) functor to (F.D.)VECT. But in the non-algebraically-closed case (or the "over a ring" case), the question is much richer, and in general there are many functors.

So in general, for any groupoid G and any commutative ring K, you can get a groupoid of symmetric monoidal (and otherwise nice) functors ${}^G\mathcal{K} \to \mathcal{K}$, where as always $\mathcal{K} = K\text{-MOD}$. If you fix G and let K vary, you get something like a "groupoid scheme", which is another word for *stack*.

With Alex Chirvasitu, we asked the opposite question: what if you fix K and vary G? The construction is obviously functorial in G, and the functor GROUPOIDS $\to$ GROUPOIDS is not obviously representable. In fact, it isn't representable at all by an honest groupoid, usually, but it is representable by a profinite groupoid (when only tested against finite groupoids). In fact, much more can be said:

Theorem: For fixed K, the representing object of the functor $G \mapsto \{$monoidal functors ${}^G\mathcal{K} \to \mathcal{K}$, monoidal natural transformations$\}$ is the étale $\pi_1(\operatorname{spec} K)$.

This is a new definition for étale π_1. We have proven the theorem as stated, i.e. for affine schemes. When $\mathcal{K} = \mathrm{QC{\small OH}}(\mathcal{O}(S)\text{-}\mathrm{{\small MOD}})$ for S a scheme, we're pretty sure the statement is true as well, but have some details to check. In fact, it should fold for (sufficiently well behaved) algebraic stacks: we've proven it for algebraic stacks of the form "affine scheme mod finite group".