Classification of $\overbrace{\quad}^{\text{semisimple}}$ TQFTs

Theo Johnson-Freyd (Dalhousie & Perimeter)
Mathematics of QFT, MPIM, 29 March 2022

Joint work in progress with David Reutter.

These slides: http://categorified.net/TQFT-MPIM.pdf

Sponsor message: https://scgcs.berkeley.edu/gcs2022-conference-and-school/

Table of contents

Cobordism Hypothesis

The cobordism hypothesis (Baez–Dolan, Lurie) asserts that (fully local, framed) TQFTs are classified by:

$$\{n\text{D TQFTs}\} = \text{Objects(the } n\text{-category of TQFTs)}.$$

The k-morphisms on this n-category of k-codimensional topological interfaces. This n-category has duals and adjoints.

The cobordism hypothesis is not contentless: It asserts that such an n-category exists! (This can fail if you allow non-local TQFTs or disallow framing-dependence.) It also says that if $\mathcal{W}$ is any n-category with duals and adjoints, then there is some restricted class of TQFTs such that $\mathcal{W} = \{\text{TQFTs in that class}\}$. But what n-category contains all TQFTs?

Semisimplicity

A $\mathbb{C}$-linear n-category $\mathcal{W}^n$ is semisimple if (cf Douglas–R):

- $\mathcal{W}^n$ is additive ($\oplus$'s) and Karoubi-complete (aka condensation-complete, cf Gaiotto–JF).
- All ($<n$)-morphisms in $\mathcal{W}^n$ have adjoints. Including 0-morphisms if $\mathcal{W}$ is monoidal.
- All hom-1-categories of $(n-1)$-morphisms are semisimple. Equivalently, all surjective n-morphisms split, and hom v-spaces of n-morphisms are f.d. Idea: adjoints $\approx$ splittings.

A semisimple n-category $\mathcal{W}^n$ is compact (cf Décoppet) if it admits a generating object. (All objects are compact projective.)

A (compact) semisimple TQFT is a TQFT valued in a (compact) semisimple symmetric monoidal n-category.

Universal target I

The universal target $\mathcal{V}^n$ for (compact) semisimple nD TQFTs is the (ind-compact) semisimple symmetric monoidal n-category such that if $\mathcal{W}^n \neq 0$ is any (compact) semisimple symmetric monoidal n-category, then $\hom_{\mathrm{sym}\otimes}(\mathcal{W}^n, \mathcal{V}^n) \neq \emptyset$.

I will take existence of $\mathcal{V}^n$ as an ansatz. In the process of computing $\mathcal{V}^n$, we will prove its uniqueness and existence.

Examples:

- Fundamental theorem of algebra: $\mathcal{V}^0 = \mathbb{C}$.
- Existence of super fibre functors (Deligne): $\mathcal{V}^1 = \mathrm{sV\small{EC}}$.

Universal target II

Notation: If $\mathcal{C}^n$ is a semisimple n-category equipped with a basepoint $1 \in \mathcal{C}^n$, then its based loop category is $\Omega\mathcal{C}^n := \mathrm{End}_{\mathcal{C}^n}(1)$. It is a monoidal semisimple $(n-1)$-category. Left adjoint to Ω is

$$\Sigma = \text{Kar(one-object delooping)} \simeq \{\text{compact projective modules}\}.$$

Lemma: $\mathcal{V}^{n-1} \simeq \Omega\mathcal{V}^n$. I.e. $\mathcal{V}^\bullet$ is a categorical spectrum.

Proof: Test universal property on $\mathcal{W}^n = \Sigma\mathcal{W}^{n-1}$.

Lemma: $(\mathcal{V}^\bullet)^\times \simeq (I\mathbb{C}^\times)^\bullet$, the dual to spheres. $\pi_0(I\mathbb{C}^\times)^n = \widehat{\pi_n\mathbb{S}}$, where $\widehat{A} := \hom(A, \mathbb{C}^\times)$. (Answers a request a Freed–Hopkins.)

Proof: Test universal property on "group algebras" of spectra.

Global categorical symmetries I

Choose an $n{+}1$D TQFT $\mathcal{Q} \in \mathcal{V}^{n+1}$. The algebra of extended operators in $\mathcal{Q}$ is

$$\mathcal{A}^n = \mathcal{Q}(S^0) = \mathrm{End}(\mathcal{Q}) \qquad \text{(inner hom in } \mathcal{V}\text{)}.$$

Can interpret $\mathcal{A}$ as a (cpt ss) $\mathcal{V}$-linear monoidal n-category.

An object of $\Omega^k \mathcal{A}^n$ is an $n{-}k$D topological defect in $\mathcal{Q}$. More precisely, if $\mathcal{Z}^{n-k} \in \mathcal{V}^{n+1-k}$ is some other $n{+}1{-}k$D TQFT, then $\hom_{\mathcal{V}}(\mathcal{Z}, \Omega^k \mathcal{A})$ is the collection of ways that an uncoupled copy of the $\mathcal{Z}$ can end along a topological defect in $\mathcal{Q}$. I.e. $\hom_{\mathcal{V}}(\mathcal{Z}, \Omega^k \mathcal{A})$ are the $n{-}k$D topological defects in $\mathcal{Q}$ which carry anomaly $\mathcal{Z}$.

Elements of $\mathcal{A}$ have adjoints $\approx$ inverses. So we also think of $\mathcal{A}$ as the global noninvertible aka categorical symmetries of $\mathcal{Q}$.

Global categorical symmetries II

Pick some other (cpt ss $\mathcal{V}$-linear) monoidal n-category $\mathcal{X}^n$, and pick a map $\mathcal{X}^n \to \mathcal{A}^n = \mathrm{End}(\mathcal{Q})$, i.e. an action of $\mathcal{X}$ on $\mathcal{Q}$ by global categorical symmetries. To gauge these symmetries requires choosing an anomaly cancellation datum, aka a fibre functor, $\mathcal{X}^n \to \mathcal{V}^n$. (Recall: $\mathcal{V}^n$ is the unit object in $\mathcal{V}^{n+1}$.)

The resulting gauged theory is

$$\mathcal{Q} /\!\!/ \mathcal{X} = \mathcal{Q} \otimes_{\mathcal{X}} \mathcal{V} \in \mathcal{V}^{n+1}.$$

Its operator content is

$$\mathcal{A} /\!\!/ \mathcal{X} \cong \mathrm{End}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{X}} \mathcal{V}) \cong \mathcal{V} \otimes_{\mathcal{X}} \mathcal{A} \otimes_{\mathcal{X}} \mathcal{V}.$$

Example: If $\mathcal{X}$ is generated by a group $\mathcal{G}$ of invertible objects, then $\mathcal{A} /\!\!/ \mathcal{X} = \mathcal{A} /\!\!/ \mathcal{G} = \mathcal{A}^{\mathcal{G}}/\mathcal{G}$.

S-matrix / Pontryagin duality

Two simple objects $X, Y \in \mathcal{A}$ are connected if $\hom(X, Y) \neq 0$.

Higher Schur's lemma (cf Douglas–R): Connectivity is an equivalence relation.

$$\pi_0\mathcal{A} := \{\text{connected components}\}. \quad \pi_k\mathcal{A} := \pi_0\Omega^k\mathcal{A}.$$

These are not typically groups. They are bases for fusion rings.

Theorem [JF–R]: If $\mathcal{A} = \mathcal{Q}(S^0)$ for an nD TQFT $\mathcal{Q}$, then there is a well-defined and invertible S-matrix with rows indexed by $\pi_k\mathcal{A}$ and columns indexed by $\pi_{n-k-1}\mathcal{A}$.

Corollary: $\mathcal{A}$ is k-connected ($\mathcal{A} = \Sigma^k\Omega^k\mathcal{A}$) iff $\Omega^{n-k-1}\mathcal{A}$ is trivial.

Lan–Kong–Wen / Surgery theory I

Suppose $\mathcal{Q} \in \mathcal{V}^{n+1}$, $\mathcal{A}^n = \mathrm{End}(\mathcal{Q})$, and $p = \lfloor \frac{n-1}{2} \rfloor$.

By the stabilization hypothesis, the E_{n+1-p}-monoidal p-category $\Omega^{n-p}\mathcal{A}$ is symmetric monoidal. By universality of $\mathcal{V}$, we can choose an anomaly cancellation datum $\Omega^{n-p}\mathcal{A} \to \mathcal{V}^p$.

In the gauged theory $\mathcal{Q} /\!\!/ \Omega^{n-p}\mathcal{A}$, we have killed all operators of high degree. By Pontryagin duality, $\mathcal{Q} /\!\!/ \Omega^{n-p}\mathcal{A}$ is p-connected.

- If $n = 2p + 1$ is odd, then $\mathcal{Q} /\!\!/ \Omega^{n-p}\mathcal{A}$ is invertible.
- If $n = 2p + 2$ is even, then $\mathcal{Q} /\!\!/ \Omega^{n-p}\mathcal{A}$ has nontrivial operators only in middle dimension.

Galois / Tannaka duality

There is a stronger statement implied by universality. Given cpt ss symmetric monoidal $\mathcal{W}^n$, define

$$\mathsf{Spec}(\mathcal{W}^n) := \mathsf{hom}_{\mathsf{sym}\otimes}(\mathcal{W}^n, \mathcal{V}^n).$$

It is a π-finite n-groupoid, with an action by the *n*-categorical absolute Galois group $\mathrm{Gal}_n := \mathsf{Aut}_{\mathsf{sym}\otimes}(\mathcal{V}^n)$.

Theorem: The canonical map

$$\mathcal{W}^n \to \{\mathrm{Gal}_n\text{-equivariant functors } \mathsf{Spec}(\mathcal{W}^n) \to \mathcal{V}^n\}$$

is an equivalence of symmetric monoidal *n*-categories.

The proof requires universality of $\mathcal{V}$, some ambidexterity (cf Hopkins–Lurie), and some connectivity results.

Suppose $n = 2p+1$ is odd. Given n+1D TQFT $\mathcal{Q}$ with operators $\mathcal{A}^n$, the p-groupoid $X = \mathrm{Spec}(\Omega^{n-p}\mathcal{A})$ is the space of anomaly cancellation data. This p-groupoid carries a bundle of invertible n+1D TQFTs (fibre is $\mathcal{Q} /\!\!/ \Omega^{n-p}\mathcal{A}$). But invertible TQFTs are classified by $\mathcal{V}^\times = I\mathbb{C}^\times$. By Tannakian duality, $\mathcal{Q}$ is recovered from this bundle. Physically, the TQFT is the sigma model with target $= X$ and Lagrangian $=$ [the class in $I\mathbb{C}^\times(X)$].

Theorem: For any cpt ss sym mon $\mathcal{W}^{n+1}$, set $G = \mathrm{Gal}(\mathcal{V}/\mathcal{W})$. The n+1D TQFTs valued in $\mathcal{W}$ are canonically classified by:

- a finite homotopy p-type X
- an action on X by G
- a G-equivariant $I\mathbb{C}^\times$-valued class on X of degree n+1.

Suppose $n = 2p+2$ is even. Then our $n+1$D TQFT $\mathcal{Q}$ is canonically the global sections of a bundle over $X = \mathrm{Spec}(\Omega^{n-p}\mathcal{A})$ of TQFTs with operators only p-dimensional operators. More precisely, for each $x \in X$, let $\mathcal{Q}_x$ be the fibre, and $\mathcal{A}_x$ its operators. Then $\pi_k \mathcal{A}_x = 0$ except for $\pi_{p+1}\mathcal{A}_x$.

Theorem: Suppose that $p \geq 1$ (i.e. $n+1 \geq 5$).

- Although typically $\pi_k \mathcal{C}$ is just a set, $A_x := \pi_{p+1}\mathcal{A}_x$ is an abelian group.
- $\mathcal{A}_x$ is recovered from A_x together with a class in $(I\mathbb{C}^\times)^{2p+2}(K(A_x, p+2))$, the set of symmetric ($p$ even) or antisymmetric (p odd) forms on A_x.
- This form gives the S-matrix, which is necessarily nondegenerate.

ENO / Postnikov extension theory I

What is the overall structure of $\mathcal{V}^n$? What is the absolute Galois group $\mathrm{Gal}_n := \mathrm{Aut}(\mathcal{V}^n)$?

The identity component is $\Sigma\Omega\mathcal{V}^n = \Sigma\mathcal{V}^{n-1}$. Its objects are those TQFTs which admit a topological boundary condition.

Proposition (cf JF–Yu): $\pi_0\mathcal{V}^n$ is an abelian group.

Corollary: $\mathcal{V}^n$ is a $\pi_0\mathcal{V}^n$-graded extension of $\Sigma\Omega\mathcal{V}^n = \Sigma\mathcal{V}^{n-1}$.

Thus we can build $\mathcal{V}^n$ by induction if we know $\mathcal{V}^{n-1}$, $\pi_0\mathcal{V}^n$, and the Postnikov aka ENO extension class

$$\pi_0\mathcal{V}^n \to (\Sigma^2\mathcal{V}^{n-1})^\times.$$

LHS is concentrated in π_0. RHS $= (I\mathbb{C}^\times)^{n+1}$ except for π_0 and π_1.

ENO / Postnikov extension theory II

Moreover: $\mathcal{V}^n$ is universal $\Leftrightarrow \pi_0\mathcal{V}^n$ represents the presheaf

$$A \mapsto \pi_0 \operatorname{hom}\big(A, (\Sigma^2\mathcal{V}^{n-1})^\times\big), \quad A \in \textsc{AbGp}.$$

Why is this representable? Use $\Sigma^2\mathcal{V}^{n-1} \approx (I\mathbb{C}^\times)^{n+1}$ and various LESs to show the required exactness.

Corollary: $\mathcal{V}^n$ exists.

Proposition: The same computation produces an exact sequence

$$\mathsf{Inv}(\Sigma\mathcal{V}^{n-1}) \to \widehat{\pi_n\mathbb{S}} \to \pi_0\mathcal{V}^n \to \mathsf{Inv}(\Sigma^2\mathcal{V}^{n-1}) \to \widehat{\pi_{n+1}\mathbb{S}}$$

where $\mathsf{Inv}(-)$ = iso classes of invertible objects. The maps to $\widehat{\pi_\bullet\mathbb{S}}$ record the partition function of the invertible TQFT.

Which invertible TQFTs are anomalies? I

$$\mathrm{Inv}(\Sigma\mathcal{V}^{n-1}) \to \widehat{\pi_n\mathbb{S}} \to \pi_0\mathcal{V}^n \to \mathrm{Inv}(\Sigma^2\mathcal{V}^{n-1}) \to \widehat{\pi_{n+1}\mathbb{S}}$$

Unpacking gives:

$$\begin{aligned}
\mathrm{Inv}(\Sigma\mathcal{V}^n) &= \text{invertible } n{+}1\text{D TQFTs which admit a topological boundary condition} \\
&= \text{anomalous } n\text{D TQFTs modulo ``remember only the anomaly''} \\
\mathrm{Inv}(\Sigma^2\mathcal{V}^{n-1}) &= \text{invertible } n{+}1\text{D TQFTs with a connected component of topological boundary conditions} \\
&= \text{anomalous } n\text{D TQFTs modulo top'l interfaces}
\end{aligned}$$

Corollary: $\mathrm{Inv}(\Sigma^2\mathcal{V}^{n-1}) \twoheadrightarrow \mathrm{Inv}(\Sigma\mathcal{V}^n) \hookrightarrow \widehat{\pi_{n+1}\mathbb{S}}$.

Which invertible TQFTs are anomalies? II

$$\cdots \to \widehat{\pi_n \mathbb{S}} \to \pi_0 \mathcal{V}^n \to \mathsf{Inv}(\Sigma^2 \mathcal{V}^{n-1}) \to \dots$$

When $n = 3$, $\Sigma^2\mathcal{V}^1 \cong \mathrm{FusCat}$ and $\Sigma^2\mathcal{V}^2 \cong \mathrm{BrFusCat}$, so:

$$\mathsf{Inv}(\Sigma^2\mathcal{V}^1) = 0, \quad \pi_3\mathbb{S} = \mathbb{Z}/24, \quad \mathsf{Inv}(\Sigma^2\mathcal{V}^2) = \mathrm{qWitt}, \quad \pi_4\mathbb{S} = 0,$$

where qWitt is the quantum Witt group of slightly-degenerate braided fusion categories studied by Davydov–Nikshych–Ostrik, and so $\pi_0\mathcal{V}^3 = (\mathbb{Z}/24).\mathrm{qWitt}$. This recovers (and was inspired by) a construction due to Freed–Scheimbauer–Teleman.

When $n \geq 4$, the method of Lan–Kong–Wen applies just as well to anomalous nD TQFTs as to absolute ones. Every even-dimensional anomalous TQFT is a sigma model, and hence nonanomalous. I.e. when n is odd,

$$\mathsf{Inv}(\Sigma^2\mathcal{V}^{n-1}) \overset{0}{\to} \widehat{\pi_n\mathbb{S}}.$$

Which invertible TQFTs are anomalies? III

Take $n > 4$ even, $\mathcal{Q}$ an anomalous $n-1$D TQFT, and surger it to one with only middle-dim operators, classified by abelian group A and an (anti)$^{n/2}$symmetric form. This has a topological boundary condition iff A admits a Lagrangian subgroup.

Corollary:

- When $n = 4k+2$, $\mathsf{Inv}(\Sigma^2\mathcal{V}^{n-1}) \cong \mathbb{Z}/2$. The map $\mathsf{Inv}(\Sigma^2\mathcal{V}^{n-1}) \to \widehat{\pi_n\mathbb{S}}$ selects the Arf–Kervaire invariant.
- When $n = 4k \geq 8$, $\mathsf{Inv}(\Sigma^2\mathcal{V}^{n-1}) \cong \mathrm{Witt}$ is the (classical) Witt group of abelian groups with a symmetric bilinear form. Since every symmetric bilinear form admits a quadratic refinement, the map $\mathsf{Inv}(\Sigma^2\mathcal{V}^{n-1}) \to \widehat{\pi_n\mathbb{S}}$ vanishes.

Corollary: The only invertible TQFTs which arise as anomalies of topological theories are (trivial and) Arf–Kervaire invariants.

The absolute Galois group

The ∞-categorical absolute Galois group is $\mathrm{Gal} := \mathsf{Aut}(\mathcal{V}^\bullet)$. Since $\mathcal{V}^n$ is a graded extension of $\Sigma\mathcal{V}^{n-1}$, find $\pi_n\mathrm{Gal} = \widehat{\pi_0\mathcal{V}^n}$. In other words, $\pi_4\mathrm{Gal} = \widehat{\mathrm{qWitt}}$ and, for $k > 1$,

$$\begin{array}{ccccccc} & & & \dots & \to & 0 \\ \to & \pi_{4k}\mathrm{Gal} & \to & \pi_{4k}\mathbb{S} & \xrightarrow{0} & \widehat{\mathrm{Witt}} \\ \to & \pi_{4k-1}\mathrm{Gal} & \to & \pi_{4k-1}\mathbb{S} & \to & 0 \\ \to & \pi_{4k-2}\mathrm{Gal} & \to & \pi_{4k-2}\mathbb{S} & \to & \mathbb{Z}/2 \\ \to & \pi_{4k-3}\mathrm{Gal} & \to & \pi_{4k-3}\mathbb{S} & \to & 0 \\ \to & \dots & & & & \end{array}$$

The right-hand column is almost L-theory. As such, Gal is almost $\mathrm{PL} = \bigcup \mathrm{PL}(n) \cong \mathsf{fibre}(\mathbb{S} \to L)$, where $\mathrm{PL}(n)$ = piecewise-linear automorphisms of $\mathbb{R}^n$. **Remark (Lurie):** $\mathrm{PL}(n) = \mathsf{Aut}(\mathrm{Bord}_n^{\mathsf{fr}})$.

There's also something funny about $\pi_0\mathrm{Gal} \stackrel{?}{=} \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$ versus $\pi_0\mathrm{PL} = \pi_0\mathrm{O} = \mathbb{Z}/2$. But that's a story for another day...

References and acknowledgements

My story is based on to-be-written joint work with David Reutter. It also draws on ideas from and conversations with Davide Gaiotto, Mike Hopkins, and Matthew Yu. Inspiration came from analyzing constructions of Lan–Kong–Wen and Freed–Scheimbauer–Teleman.

Some pieces of the story can be found in my older papers:

1507.06297 Spin, statistics, orientations, unitarity. *AGT*.

2010.07950 Fusion 2-categories with no line operators are grouplike. With M. Yu. *Bull. Aust. Math. Soc.*

2003.06663 On the classification of topological orders. *CMP*.

2104.04534 Topological orders in (4+1)-dimensions. With M. Yu.

SCGCS PERIMETER INSTITUTE DALHOUSIE UNIVERSITY NSERC CRSNG