Deformation Quantization
Exercise Sheet 1

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[!] *marks a more important exercise*

Exercise 1 (Poisson brackets from deformations)**.** [!] Let us reverse the logic and obtain a Poisson bracket from a deformation of a commutative algebra. That is, consider

$$m = m_0 + \hbar m_1 + \hbar^2 m_2 + \dots \quad : A \otimes A \to A$$

such that (A, m_0) is a commutative algebra and m is associative. Show that

$$\{a, b\} = \frac{m_1(a,b) - m_1(b,a)}{2}$$

is a Poisson bracket.

Exercise 2 (Wigner-Weyl transform)**.** For a function $a(p,q)$ on $\mathbb{R}^2$, the corresponding operator $W(a)$ can be defined by its action on wavefunctions $u(q)$ as

$$W(a) : u(q) \mapsto \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{\frac{i}{\hbar} p(q-q')} a\left(\frac{q+q'}{2}, p\right) u(q')\, dq'\, dp. \tag{1}$$

1. The scalar product of wave functions u is

$$\langle u_1 | u_2 \rangle := \int_{\mathbb{R}} \overline{u_1(q)} u_2(q)\, dq.$$

Show that the transpose w.r.t. this scalar product satisfies

$$W(a)^\dagger = W(\overline{a}).$$

2. Let $a(p,q)$ be a linear function on the phase space. Use the formula for the Moyal star product to show that

$$W(a^n) = \underbrace{W(a) \circ \dots \circ W(a)}_{n \text{ times}}$$

Conclude that “W(monomial) =average of all permutations of $\hat{p}$s and $\hat{q}$s” holds.

3. [!] Again using Moyal product, prove Wick’s theorem:

$$W(z_1, \dots, z_n) = \hat{z}_1 \cdots \hat{z}_n + \sum_{k \geq 1} \frac{1}{2^k} \sum_{(i_1, j_1) \dots (i_k, j_k)} [\hat{z}_{i_1}, \hat{z}_{j_1}] \cdots [\hat{z}_{i_k}, \hat{z}_{j_k}] \cdot (\hat{z}\text{s without the contracted variables})$$

(z is p or q). In other words, we sum over all contractions of pairs by the commutator.

Start by finding a recursion $W(zF) = \hat{z}\hat{F} +$ terms. Optionally, you can go the full circle and recover the Moyal product from Wick’s theorem.

4. Find the $\hbar^3$ correction for Liouville’s equation.