Deformation Quantization
Exercise Sheet 3

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! *marks a more important exercise*

Exercise 1 (Jacobi in three coats)**.** Recall that the Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ can be encoded in a "bivector" $\pi = \pi^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$, $\pi^{ij} = \{x^i, x^j\}$

1. ! Show that Jacobi for $\{\cdot,\cdot\}$ is equivalent to $[\pi,\pi] = 0$.

   Use

   - $[f, X] = -X(f)$ for X a vector field and f a function
   - graded Jacobi identity
   - $[g,[f,\pi]] = \{f, g\}$

If $\{\cdot,\cdot\}$ comes from a non-degenerate 2-form ω (via $\{f,g\} = X_f g = i_{X_f}\omega$), show that Jacobi for $\{\cdot,\cdot\}$ is equivalent to closedness of ω. Use the Cartan magic formula

$$\mathcal{L}_X = i_X d + d i_X$$

and also

$$\mathcal{L}_X i_Y = i_{[X,Y]} + i_Y \mathcal{L}_X.$$

Exercise 2 (Maurer-Cartan elements)**.**

- T_{poly}:
  1. Convince yourselves that one can identify formal Poisson bivector fields $\pi \in T^1_{poly}(M)[[\hbar]]$ with MC elements in $\hbar T_{poly}(M)[[h]]$
  2. Describe gauge-equivalence geometrically.
- D_{poly}:
  3. ! Again, convince yourselves that one can identify star products deforming m_0 with MC elements in $\hbar D_{poly}(M)[[h]]$.
  4. Prove that the notion of gauge equivalence of these MC elements coincides with our notion of equivalence of star products.

     Hint: write the MC gauge equivalence ODE for the sum

     $$m = m_0 + \underbrace{(\hbar m_1 + \hbar^2 m_2 + \ldots)}_{\text{MC element}}$$

     Show that this ODE is satisfied by $g_t \circ m \circ g_t^{-1} \otimes g_t^{-1}$ for $g_t = e^{t\lambda}$, $\lambda \in \hbar D^0_{poly}(M)[[h]]$.