Deformation Quantization

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Contents

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The land on which UNB is situated is the unceded and unsurrendered territory of the Wolastoqey (Maliseet)[1], whose territories are governed by the Peace and Friendship Treaties[2], which recognized Wolastoqey title, and established rules for what was to be an ongoing relationship with the Nations.

Disclaimer (inspired by Theo's[3]): As is the nature of notes, the information Ján communicated in lecture has only been encoded in these notes after being processed in the black box that is my brain. Anything you find helpful or creative or insightful is due to Ján, and anything you find misleading or incorrect is due to me. Please let me know[4] if you find errors/corrections!

(comment by Jan) I am very grateful to Tian for capturing the lectures this way. I went over the notes and made small changes to improve clarity and to remove some errors which appeared in the lectures. The lecture series is complemented by a set of four exercise sheets which you can find on the school website. There are solutions available upon request at jan.pulmann@gmail.com. Finally, I didn't spend much time on attributing the results presented. Therefore, at the end of these notes, you can now find an annotated reference list.

[1] https://wnnb.wolastoqey.ca/about-us/our-history/

[2] https://www.rcaanc-cirnac.gc.ca/eng/1100100028589/1539608999656

[3] https://categorified.net/otherdocs.html

[4] tiand@umich.edu

Day 1: physical motivation

DAY 1

We'll talk about the physical motivation of deformation quantization (how it comes from quantum mechanics).

SECTION 1.1
What is deformation quantization?

def 1.1 A *Poisson algebra* $(A, m_0, \{\})$ is a commutative algebra (with multiplication m_0 and unit u) with a Poisson bracket

$$\{\cdot,\cdot\} : A \otimes A \to A$$

which is

- antisymmetric,
- a derivation in each argument, i.e. for $a, b \in A$:

$$\{ab, c\} = a\{b, c\} + \{a, c\}b, \qquad \{a, bc\} = \{a, b\}c + b\{a, c\}$$

- satisfies the Jacobi identity:

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$$

def 1.2 Let A be a Poisson algebra. A *deformation quantization* of $(A, m_0, \{\})$ is an $\hbar$-dependent map $m : A \otimes A \to A$ such that

- we can recover m_0

$$\lim_{\hbar \to 0} m = m_0$$

- the old unit is still a unit

$$m(u, a) = m(a, u) = a$$

- commutativity is deformed (we'll drop the $-i$ factor after this lecture)

$$\lim_{\hbar \to 0} \frac{m(a,b) - m(b,a)}{-i\hbar} = \{a, b\}$$

- m is associative.

We'll write $m(a,b) = a \star b$ and call it the *star product*.

q 1.3 What about the next order?
Ján: We don't add new constraints. See exercise

q 1.4 Is $\hbar$ a real parameter?
Ján: It doesn't have to be. it can just be a formal variable so that m is formal power series in $\hbar$, and we take things order-by-order. It's harder (but possible) to make m actually suitably differentiable in $\hbar$ so that this limit is an actual limit.

Still some things are unclear:

- why do we break commutativity and keep associativity?
- why is this called quantization?

SECTION 1.2

QM in phase space

Basic idea: there is an isomorphism W between operators in QM and functions $a(q,p)$ on the phase space (the space you need to specify initial conditions of a particle, i.e. position and momenta). Today, we'll take our phase space to be $\mathbb{R}^2$ with Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}.$$

The star product corresponds to operator composition under this isomorphism W. That is, for functions a, b on the phase space

$$a \star b = W^{-1}(W(a) \circ W(b)).$$

In 1d QM:

- states are (square-integrable) complex functions on $\mathbb{R}$. Up to some identification, these form the Hilbert space $\mathcal{H}$ of the theory

- observables (things we can measure) are self-adjoint linear operators on $\mathcal{H}$ (which are denoted with hats). For example,
  - we have the position operator $(\hat{q}U)(q) = qU(q)$
  - and a momentum operator $(\hat{p}U)(q) = -i\hbar\frac{\partial u}{\partial q}(q)$

  and $[\hat{q}, \hat{p}] = i\hbar\,\mathrm{id}_{\mathcal{H}}$.

Note that these operators are unbounded and not defined on arbitrary square integrable functions, but we ignore these subtleties here.

q 1.5 **Luuk:** This is *not* an isomorphism of algebras?
Ján: Yes. It is not when we use the multiplication m_0 (phase space is commutative while operators are not), but it *is* when we use m (i.e. star product).

def 1.6 (Weyl '27) W takes polynomial functions $q^k p^l$ to the average of all possible ways of ordering $\hat{p}$s and $\hat{q}$s.

ex 1.7
- $W(p^l) = \hat{p}^l$, $W(q^k) = \hat{q}^k$
- $W(qp) = \frac{1}{2}(\hat{q} \circ \hat{p} + \hat{p} \circ \hat{q})$
- $W(p^2 q) = \frac{1}{3}(\hat{p}^2\hat{q} + \hat{p}\hat{q}\hat{p} + \hat{q}\hat{p}^2)$

W has some nice properties:

- it sends

$$(\lambda q + \mu p)^n \longmapsto (\lambda\hat{q} + \mu\hat{p})^n$$

  since as a product, the term with coefficient $\lambda^k\mu^{n-k}$ is already the sum of all possible ways of ordering p's and q's, i.e. the terms are already symmetrized.

- it intertwines complex conjugation and Hermitian adjoint (see exercises).

We'll use an alternative formula: for $a(q, p) \in O(\mathbb{R}^2)$ (the algebra of functions on $\mathbb{R}^2$), define $\hat{a} := W(a)$. We'll compute

$$(\hat{a}u)(q) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} e^{\frac{i}{\hbar}p(q-q')} a\left(\frac{q+q'}{2}, p\right) dp\, u(q')\, dq'$$

which is sometimes called the *Wigner–Weyl transform*. Convergence is okay as long as a is a polynomial, otherwise we want a which tends quickly enough to zero away from the origin. Using this formula we can define the *integral kernel* of $\hat{a}$

$$(\hat{a}u)(q) =: \int K_{\hat{a}}(q, q')u(q')\, dq'$$

i.e. $K_{\hat{a}} = \langle q|\hat{a}|q'\rangle$; explicitly

$$K_{\hat{a}}(q, q') = \frac{1}{2\pi\hbar}\int_{\mathbb{R}} e^{\frac{i}{\hbar}p(q-q')} a\left(\frac{q+q'}{2}, p\right) dp.$$

Now, by making the substitution

$$K_{\hat{a}}(q + \frac{t}{2}, q - \frac{t}{2}) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}} e^{\frac{i}{\hbar}t} a(q, p)\, dp$$

def 1.8 (Wigner '32) This is the inverse of W!

$$a(q, p) = \int K_{\hat{a}}\left(q + \frac{t}{2}, q - \frac{t}{2}\right) e^{-\frac{i}{\hbar}pt}\, dt =: W^{-1}(a) \tag{1.1}$$

Now, we want to compute $a \star b = W^{-1}(\hat{a} \circ \hat{b})$.
The integral kernel is not difficult to write:

$$K_{\hat{a}\circ\hat{b}}(q, q'') = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^3} e^{\frac{i}{\hbar}p_1(q-q')+\frac{i}{\hbar}p_2(q'-q'')} \times a\left(\frac{q+q'}{2}, p_1\right) b\left(\frac{q'+q''}{2}, p_2\right) dp_1\, dp_2\, dq'$$

Then using eq. (1.1),

$$(a \star b)(q, p) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^4} \exp\left(\frac{i}{\hbar}\left[p_1\left(q - q' + \frac{t}{2}\right) + p_2\left(q' - q + \frac{t}{2}\right) - pt\right]\right) \times a\left(\frac{q + \frac{t}{2} + q'}{2}, p_1\right) b\left(\frac{q' + q - \frac{t}{2}}{2}, p_2\right) dp_1\, dp_2\, dq'\, dt.$$

Now, looking at the arguments of a and b, we introduce the substitution $\frac{q+t/2+q'}{2} = q + \frac{q'+t/2-q}{2} =: q + u$ and $\frac{q'+q-t/2}{2} = q + \frac{q'-t/2-q}{2} = q + v$, we can rewrite the exponent as $-2p_1 v + 2p_2 u - 2p(u - v)$. Then, changing $p_1 = s + p$ and $p_2 = r + p$, we obtain:

$$(a \star b)(q, p) = \frac{1}{(\pi\hbar)^2}\int_{\mathbb{R}^4} e^{\frac{2i}{\hbar}[r\cdot u - s\cdot v]} \times a(q + u, p + s) b(q + v, p + r)\, ds\, dr\, du\, dv$$

Then, writing $x = (q, p)$, $y = (u, s)$, $z = (v, r)$ (all vectors in $\mathbb{R}^2$), we obtain:

$$(a \star b)(q,p) = \frac{1}{(\pi\hbar)^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \exp\left(\frac{i}{\hbar}\left(y^T\ z^T\right)\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} y \\ z\end{pmatrix}\right) \\ \times\ a(x+y)b(x+z)\,dy\,dz$$

To perform this integral, we just notice it computes moments of a Gaussian integral. It is thus given by inverting the 4×4 matrix, contracting it with derivatives in the exponenetial, and letting it act on $a(x+y)b(x+z)$. Writing $y = (q_y, p_y)$ and $z = (q_z, p_z)$, we get the *Moyal product*.

$$(a \star b)(q,p) = e^{\frac{i\hbar}{2}\left(\partial_{q_y}\partial_{p_z} - \partial_{p_y}\partial_{q_z}\right)} a(x+y)b(x+z)$$

ex 1.9 This tells us that

$$q \star p = qp + \frac{i\hbar}{2},$$

which can be used to check we got the correct sign.

SECTION 1.3

Summary

In QM using W^{-1}, we found a correspondence between:

$$\begin{aligned} \text{operators} &\longrightarrow \text{functions on phase space} \\ \text{composition } \circ &\longrightarrow \star \\ \text{states } |\psi\rangle \to |\psi\rangle\langle\psi| &\longrightarrow \text{quasi probability distributions } O(\mathbb{R}^2) \\ \text{Schrödinger equation} &\longrightarrow \hat{\rho} = \frac{1}{i\hbar}(H \star \rho - \rho \star H) \\ \text{expectation values } \langle \hat{a} \rangle_\rho &\longrightarrow \int \rho(q,p) a(q,p)\,dp\,dq \end{aligned}$$

q 1.10 **Sal:** how does this map depend on pure/mixed state?
Ján: no. you can detect pure states with the same criteria as you do typically (square of density matrix has trace one).

q 1.11 **Luuk:** why are these called "quasi-probability distributions"?
Ján: they're defined by the image of this map. they look like probability distributions, but they can be negative (which in some ways captures the quantumness). if you convolve with something the size of $\hbar$, they become positive. they satisfy an equation which

looks like Liouville equation with corrections.
Kyle: we only know things about the moments of the distribution, but this only tells us about sufficiently nice distributions.
Luuk: but these are positive operators since they come from states?
Kyle: yes. but they're not positive in the sense of functions valued at points.
Ján: there was a "big fight" between Dirac and Moyal because Dirac was convinced it's impossible to formulate distributions on this phase space due to negativity.

q 1.12 **Nian:** are the quantization methods the same? how do they differ? [there were questions about quantizations in the sense of orderings, other than Weyl ordering.]
Ján: They (the star products) are equivalent on $\mathbb{R}^n$. However, on manifolds with differing second cohomology class, the quantization procedures are no longer equivalent.
Me: is this physically relevant?
Ján: In the case of deformation quantization, not really. Deformation quantization is kind of a formal procedure ($\hbar$ is not really a number, and the deformation is a formal power series in $\hbar$).

Day 2

DAY 2

From now on, we'll take $-i\hbar \to \hbar$. We'll also take $\hbar$ as a formal parameter and think about *formal* deformation quantizations.

def 2.1 Let $(A, m_0, \{\})$ be a Poisson algebra with unit u (over a field $\Bbbk$). A *formal deformation quantization* is given by a sequence of maps $m_i : A \otimes A \to A$ for $i \geqslant 0$ such that

- m_0 is as above,
- $m_1(a, b) - m_1(b, a) = \{a, b\}$,
- $m_i(u, a) = m_i(a, u) = 0$ for all $i > 0$

and such that

$$m = \sum_{i \geqslant 0} h^i m_i$$

is an associative product on $A[\![\hbar]\!]$.

Two such deformation quantizations of the same underlying Poisson algebra are *equivalent* if there is a sequence of maps $g_i : A \to A$ for $i \geqslant 0$ such that:

- $g_0 = \mathrm{id}_A$, and
- $(A[\![\hbar]\!], m) \xrightarrow{g:=\sum_i g_i \hbar^i} (A[\![\hbar]\!], m')$ is a map of algebras.

q 2.2 Should we require isomorphism?
Ján: Since $g_0 = \mathrm{id}_A$, this is automatic: we have an algebra homomorphism of which the underlying map is an isomorphism of vector spaces.

Then:

$$m' \circ (g \otimes g) = g \circ m$$

implies

$$(m_0 + \hbar m_1' + \dots) \circ (\mathrm{id} + \hbar g_1 + \dots) \otimes (\mathrm{id} + \hbar g_1 + \dots) \\ = (\mathrm{id} + \hbar g_1 + \dots) \circ (m_0 + \hbar m_1 + \dots).$$

Equating at each order gives:

$$\hbar^0 : m_0 = m_0$$
$$\hbar^1 : m_0 \circ (\mathrm{id} \otimes g_1 + g_1 \otimes \mathrm{id}) + m_1' = m_1 + g_1 \circ m_0$$

from the second relation, we see that the antisymmetrization $\{\cdot,\cdot\} = \{\cdot,\cdot\}'$ is an invariant of this equivalence relation.

q 2.3 Why is this true?
Ján: antisymmetrize the equation, then since we have m_0 (commutative), the terms with g_1 need drop out.

remark

- if $\hbar$ is not formal, one speaks of "strict deformation quantization"
- if A is an algebra of functions on a manifold, one almost always adds the requirement that m_i, g_i are differential operators (without a constant term), i.e. in coordinates,

$$m_i(a, b) = \sum_{IJ} m_i^{IJ}(x)\, \partial_I a\, \partial_J b$$

where I, J are multi-indices. Note that if these are differential operators without a constant term, the condition $m_i(a, u) = m_i(u, a)$ is automatically satisfied.

def 2.5 (alternative definition) A *formal deformation quantization* of $(A, m_0, \{\})$ is an associative $\Bbbk[[\hbar]]$-linear algebra $(A_\hbar, m)$ together with an isomorphism of $\Bbbk[[\hbar]]$-modules called a *section* $\varphi : A[[\hbar]] \to A_\hbar$ such that for $a, b \in A$

- φ is an algebra isomorphism modulo $\hbar$
- $\{a, b\} = \frac{1}{\hbar}(\varphi(a) \star \varphi(b) - \varphi(b) \star \varphi(a)) \mod \hbar$
- $\varphi(u) = u_{A_\hbar}$

Often we have a nice construction of $A[[\hbar]]$ as an abstract power series, but the choice of such an isomorphism involves things up (see Skein modules).

SECTION 2.1

Problem of deformation quantization

Given a Poisson algebra $(A, m_0, \{\})$, we want to find all formal deformation quantizations up to equivalence.

Let's try to deformation quantize directly. Given $(A, m_0, \{\})$, let's look for $m_1, m_2, \dots$. From associativity at order $\hbar^1$:

$$m_0(m_1 \otimes 1) + m_1(m_0 \otimes 1) - m_0(1 \otimes m_1) - m_1(1 \otimes m_0) = 0.$$

If $m_1 = \{,\}$ this is automatically satisfied[1].

Then for $m_0 + \hbar m_1$, associativity is ok up to $\hbar^1$, and we want to add $\hbar^2 m_2$

$$\begin{aligned} m_2(m_0 \otimes 1) + &m_1(m_1 \otimes 1) + m_0(m_2 \otimes 1) \\ &- m_2(1 \otimes m_0) - m_1(1 \otimes m_1) - m_0(1 \otimes m_2) = 0 \end{aligned}$$

For higher powers of $\hbar$, this gets even worse, which motivated the following:

def 2.6 (Hochschild cochain complex) For an algebra $(A, m_0(a, b) = ab)$, we define a cochain complex $C^n(A) = \mathrm{Hom}_{\mathrm{Vect}}(A^{\otimes n}, A)$ with a differential

$$\begin{aligned} (\delta f)(a_0, \dots, a_n) = &a_0 f(a_1, \dots, a_n) - f(a_0 a_1, \dots, a_n) + \cdots \\ &+ (-1)^n f(a_0, \dots, a_{n-1} a_n) + (-1)^{n+1} f(a_0, \dots, a_{n-1}) a_n \end{aligned}$$

Claim: $(C^n(A), \delta)$ is a cochain complex (see exercises).

[1] Leibniz rule

In general, extending $m_0 + \cdots + \hbar^k m_k$ to an associative product 1 power of $\hbar$ higher using $\hbar^{k+1} m_{k+1}$ gives an equation

$$\delta m_{k+1} = \sum_{\substack{i+j=k+1 \\ i,j \geqslant 1}} m_i(m_j \otimes 1 - 1 \otimes m_j) \tag{2.1}$$

Note that this takes three inputs so it takes place in $C^3(A)$. We can thus reformulate the problem of extending m cohomologically: it is possible to find m_{k+1} if and only if the right hand side of Equation (2.1) is exact. If $H^3(A) = 0$, it's always exact. Otherwise, we need more work.

remark **Ján:** Changing m using a gauge transformation $(\mathbb{1} + \hbar^{k+1} g_{k+1} \in C^1(A))$ changes $m_{k+1} \to m_{k+1} + \delta g_{k+1} \in C^2(A)$ by an exact term. So $H^2(A)$ classifies possible extensions at each order. That is, $H^2(A)$ is the space of solutions up to equivalence of eq. (2.1) (if a solution exists).

q 2.8 Is it typical to go level by level?
Ján: Yes. When we go level by level, we get a linear equation for the next level.

q 2.9 It seems that most Poisson algebras are deformable.

remark We've ignored units again. We can define a vector subspace (which is actually a subcomplex) which vanishes on units (v.o.u) and moreover if $A = C^\infty(M)$ for smooth manifold M:

$$C^n_{\text{dif. op}}(A) \subset C^n_{\text{v.o.u}}(A) \subset C^n(A)$$

which induce isomorphisms on cohomology.

SECTION 2.2

Simple examples

ex 2.11 Take $A = C^\infty(\mathbb{R}^n)$, $\{f, g\} = \sum_{ij} \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$ where π^{ij} are constants. Then

$$f \star g = f e^{\frac{\hbar}{2} \overleftarrow{\partial_i} \pi^{ij} \overrightarrow{\partial_j}} g$$

where the differential operators with arrows act in the direction of the arrows (i.e. ∂_i acts on f and ∂_j acts on g). This is the (Poisson case of the) *Moyal $\star$-product*.

ex 2.12 Let $\mathfrak{g}$ be a Lie algebra. Then we can define a Poisson algebra on $\operatorname{Sym} \mathfrak{g}$ on the generators (and extend by derivation) $\{x, y\} = [x, y]$ for

$x, y \in \mathfrak{g} \subset \operatorname{Sym} \mathfrak{g}$.
Then the quantization $U\mathfrak{g}$ is called a *filtered deformation quantization* (to get back $\hbar$ and obtain a formal deformation quantization, see exercise).

ex 2.13 (quantum torus) This is an example of deformation quantization in the sense of the "alternative definition". The algebra of functions on the quantum torus is generated by $X^{\pm 1}, Y^{\pm 1}$ such that $XY = qYX$ where $q = e^{\hbar}$ give formal deformation quantization. This quantizes $\mathcal{O}(\mathbb{R}^\times \times \mathbb{R}^\times)$ with $\{x, y\} = xy$. You have to find a section to get a honest formal deformation quantization, see exercises.

q 2.14 **Kabir:** Can these be thought of as operators?
Ján: Yes, by taking $X = e^{i\hat{q}}$ and $Y = e^{i\hat{p}}$, the BCH formula gives us the commutation relation $XY = qYX$.

Day 3

DAY 3

SECTION 3.1

Summary

First lecture: physical motivation of star products from QM as functions on phase space.

Second lecture: at the end we saw a few examples of Poisson brackets which are constant, linear, quadratic. Before, we saw how to extend star product one power of $\hbar$ higher, and we saw Hochschild cochain complex.

Today we'll see two important examples of Poisson algebras and important results about their deformation quantization.

SECTION 3.2

Two sources of Poisson algebras

Functions on (symplectic manifolds $\subset$ Poisson manifolds)

def 3.1 A *symplectic manifold* is a (smooth) manifold M with a nondegenerate closed 2-form ω. For $f \in C^\infty(M)$, the *Hamiltonian vector field* of f, called X_f, satisfies

$$\omega(X_f, -) = i_{X_f}\omega = -df.$$

2-forms have an associated antisymmetric matrix, and we require that this matrix is nondegenerate

Then we define $\{f, g\} := X_f g$ for $f, g \in C^\infty(M)$.

ex 3.2 ($\mathbb{R}^{2n}$) We'll call the coordinates $p, q \in \mathbb{R}^{2n}$ so that

$$\omega = \sum_i dp^i \wedge dq_i.$$

Time evolution is governed by

$$X_H f = \{H, f\} = \frac{df}{dt}$$

where H is the Hamiltonian of some system with phase space the symplectic manifold M.

thm 3.3 (Darboux) Each $m \in M$ has a neighborhood symplectically isomorphic to an open subset of $(\mathbb{R}^{2n}, \omega)$

so locally, symplectic manifolds are not interesting, unlike Riemannian manifolds

ex 3.4

- The cotangent bundle of (any) manifold T^*N is symplectic. Here q^i are coordinates on N and p_i coordinates on the fiber $\alpha = \sum_i p_i(\alpha)\, dq^i \in T^*_m N$ for $m \in N$.
- symplectic leaves of Poisson manifolds.

q 3.5 Does N have to be even dimensional?
Ján: No. It's true that symplectic manifolds are even-dimensional, but the cotangent bundle of any manifold is always even dimensional.

SUBSECTION 3.2.1

Quantizations

thm 3.6 (de Wilde, Lecomte) All symplectic manifolds admit a deformation quantization.

An easier version of this theorem to work with is below.

thm 3.7 (Fedosov, Nest–Tsygan, Deligne, Bertelson–Cohen–Gutt) Equivalence classes of $\star$ products quantizing ω are in (explicit) bijection with elements of

$$\underbrace{\frac{\omega}{\hbar} + H^2(M, R)[[\hbar]]}_{\text{characteristic class of } \star}.$$

Corollary 0.1. $\mathbb{R}^{2n}$ *has a unique* $\star$ *product.*

q 3.8 **Theo:** does Fedosov's result require compactness? [since constructing deformation as a high tensor power limit of geometric quantization would suggest you need compactness]
Ján: Fedosov is not taking the limit of geometric quantization, he's doing pointwise deformation quantization and then globalizing it. *(the answer is he doesn't require compactness)*

SUBSECTION 3.2.2

Poisson manifolds

def 3.9 A *Poisson manifold* is a manifold M equipped with a Poisson bracket on $C^\infty(M)$

$$\{f, g\} = \sum_{ij} \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

where $\pi^{ij} = \{x^i, x^j\}$.

ex 3.10

- Symplectic manifolds (those where π^{ij} is invertible), and we can construct the $\{,\}$ from the symplectic structure as above.
- $\mathfrak{g}^*$ for $\mathfrak{g}$ a Lie algebra. Note that polynomial functions on $\mathfrak{g}^*$ are just symmetric tensors of $\mathfrak{g}$. The Poisson bracket is called Kirillov–Konstant–Souriau, i.e. $\{\}_{KKS}$.
- G where $\mathfrak{g}$ is a Lie bialgebra.
- Given some surface Σ^2 and Lie group G, the moduli space of flat connections on Σ, i.e. the quotient of space of group homomorphisms

  $$\mathrm{Hom}_{\mathrm{Grp}}(\pi_1(\Sigma, x), G)/G$$

  (the group action is by conjugation). One needs to choose $t \in \mathrm{Sym}^2(\mathfrak{g})^{\mathfrak{g}}$. The quotient might be singular, so strictly speaking is not always a manifold. The quotient can be removed if $\mathfrak{g}$ has a classical r-matrix.

q 3.11 **Sophia:** Where does the t show up?
Ján: The space looks like G to some power, so to find the Poisson bracket, you need to act with correct factors using t. Alternatively, this is the phase space of Chern–Simons theory, and the t appears in the action as the level.

SECTION 3.3

Moving toward Kontsevich's theorem

There will be two main players: first the Hochschild cochain complex (from last time), and also the complex for polyvector fields.

def 3.12 The *space of polyvector fields* is given by (for $n \geqslant -1$)

$$T^n_{\text{poly}}(M) := \Gamma(M, \Lambda^{n+1}TM)$$

ex 3.13 On a Poisson manifold, $\{\cdot,\cdot\}$ can be encoded in a "bivector field π" (i.e. $n = 1$ so that we have $\pi \in \Lambda^2 TM$)

$$\pi = \pi^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \in T^1_{\text{poly}}(M)$$

Additionally, T^0_{poly} are *vector fields* and T^{-1}_{poly} are functions on M. $T^\bullet_{\text{poly}}(M)$ is a graded Lie algebra! For X, Y vector fields the *Schouten bracket* satisfies

$$[X, Y]_{Sch} = [X, Y] \qquad\qquad [X, f]_{Sch} = X(f)$$

Then for $X \in T^{|X|}_{\text{poly}}$ and $Y \in T^{|Y|}_{\text{poly}}$

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{|X|(|Y|-1)} Y \wedge [X, Z].$$

lem 3.14 The Jacobi identity for the Poisson bracket $\{f, g\}$ is equivalent to $[\Pi, \Pi]_{\text{Sch}} = 0$.

Note that this is not trivially zero because the Lie bracket is graded-antisymmetric.

From yesterday's exercises, we saw that the Hochschild cochain complex also has a Lie bracket.

Let the *polydifferential operators*

$$D^n_{\text{poly}}(M) = \underbrace{\text{Hom}(C^\infty(M)^{\otimes n+1}, C^\infty(M))}_{\text{differential operators in each argument}}$$

with $[m_0, -] = \pm\delta$ (Hochschild differential) and $[\cdot,\cdot]_{\text{Gerst}}$. The *Hochschild–Kostant–Rosenberg* map takes:

$$T^\bullet_{\text{poly}}(M), 0, [.,.]_{\text{Sch}} \xrightarrow{U_1} D^n_{\text{poly}}(M), [m_0, -], [\cdot,\cdot]_{\text{Gerst}}.$$

where:

$$U_1(X_0 \wedge \cdots \wedge X_n)(f_0, \ldots, f_n) = \frac{1}{(n+1)!}\sum_{\sigma \in S_{n+1}} (-1)^\sigma \prod_i X_{\sigma(i)}(f_i)$$

thm 3.15 (HKR) U_1 is a quasi-isomorphism, i.e. it's compatible with differentials and induces an isomorphism on homology.

Note that this is not a map of Lie algebras!

def 3.16 A *Maurer–Cartan element* of a differential graded Lie algebra $(\mathfrak{g}, d, [.,.])$ is an element $\alpha \in \mathfrak{g}^1$ such that:

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

A *gauge equivalence* between α, α' (two MC elements) is an element $\lambda \in \mathfrak{g}^0$ such that for a time-dependent solution $\alpha(t) \in \mathfrak{g}^1$

$$\partial_t \alpha(t) = d\lambda + [\alpha(t), \lambda]$$

where $\alpha(0) = \alpha$ one has $\alpha(1) = \alpha'$

Since U_1 is not a map of Lie algebras, the MC elements don't immediately match on both sides, but Kontsevich found a relationship between them which we'll see tomorrow.

q 3.17 **Sal:** How does this connect to the broader story?
Ján: The point of today was to present two important examples of deformation quantization. As we go, we encounter more and more difficult constructions. We started with universal enveloping algebra/quantum torus. Now we're thinking about the pinnacle of the program of dq, which is dq of Poisson manifolds.

q 3.18 **Branimir:** Suppose we have a Poisson bracket on manifold. The Moyal picture suggests we can exponentiate and get a star product. What stops us from doing this in general?
Ján: if you try to prove that Moyal product is associative with Poisson bracket, we need to commute all derivatives through the exponent. This is not possible when the exponent is not constant (is a function). If you have a locally constant Poisson structure, then this is okay. The formula for the Kontsevich star product uses derivatives of the Poisson bracket.

q 3.19 **Cormack:** Does H^3 vanish on Poisson manifold?
Ján: Not necessarily. From U_1 it's the space of 3-vector fields.

q 3.20 **Luuk:** We concluded that H^2 gives number of ways to extend to the next level. Does this agree with the classification result for deformation quantization of Poisson manifolds?
Ján: *(answered later in a discussion with Luuk) at each step, there is a choise of m_{k+1} parametrized by second cohomology. However, not all such choices can be extened, so the classification will*

be governed by a smaller space than bivectors/all 2-forms. For example, we can choose any biderivation to extend m_0, but at $\hbar^2$ we learn that its antisymmetric parts needs to satisfy Jacobi.

Day 4

DAY
4

SECTION 4.1

Deformation quantization of Poisson manifolds

Recap: we defined $T_{\text{poly}}(M), 0, [.,.]_{\text{Sch}}$ where the bivector $\pi = \hbar(\pi_0 + \hbar\pi_1 + \hbar^2\pi_2 + \dots) \in MC(\hbar T_{\text{poly}}(M)[\![\hbar]\!])$ (i.e. $[\pi, \pi] = 0$).

Gauge equivalence $\pi \sim \pi'$ iff there exists a vector field $X = \hbar(X_0 + \hbar X_1 + \dots)$ such that $e^{\mathcal{L}_X}\pi = \pi'$ (i.e. are determined by the flow of vector fields).

Then, we found a chain map U_1 which is an isomorphism on cohomology to $D_{\text{poly}}(M), [m_0, .]_G, [.,.]_G$. From Kontsevich, we know that U_1 is the first component of an L_∞ morphism, i.e. a morphism which takes MC elements to MC elements.

MC elements in $\hbar D_{\text{poly}}[\![\hbar]\!]$ are $\hbar m_1 + \hbar^2 m_2 + \dots$ giving a star product $m_0 + \hbar m_1 + \dots$ and gauge transformations are isomorphisms of star products.

def 4.1 An L_∞ *morphism* $U_\bullet : (\mathfrak{g}_1, d_1, [,]_1) \to (\mathfrak{g}_2, d_2, [,]_2)$ is a sequence of maps

$$U_k : \Lambda^k \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2 \qquad \text{degree } 1-k \quad (k = 1, 2, \dots)$$

where

$$d_2 U_1 = U_1 d_1,$$
$$U_1([\alpha, \beta]_1) - [U_1(\alpha), U_1(\beta)] = d_2 U_2(\alpha, \beta) - U_2(d_1\alpha, \beta) - (-1)^{|\alpha|} U_2(\alpha, d\beta).$$

In general

$$
\begin{aligned}
dU_n(\alpha_1, \dots, \alpha_n) - \sum_{k=1-n} \pm U_n(\dots, d\alpha_k \dots) \\
= -\frac{1}{2} \sum_{\substack{k,l \geqslant 1 \\ k+l=n}} \frac{1}{k!l!} \sum_{\sigma \in S_n} \pm \big[U_k(\alpha_{\sigma(1)}, \dots \alpha_{\sigma(k)}), U_l\big(\alpha_{\sigma(k+1)}, \dots \alpha_{\sigma(n)}\big)\big] \\
+ \sum_{i<j} \pm U_{n-1}([\alpha_1, \alpha_j], \text{rest of } \alpha\text{s})
\end{aligned}
$$

prop 4.2 Let $\alpha \in \mathfrak{g}_1$ be a MC element. Then $\tilde{\mathcal{L}} = U_1(\alpha) + \frac{1}{2}U_2(\alpha, \alpha) + \frac{1}{3!}U_3(\alpha, \alpha, \alpha) + \dots$ is a MC element for $\mathfrak{g}_2$.
Additionally, if U_1 is a quasi-isomorphism, then we get an bijection between gauge equivalence classes of MC elements $U_\bullet : MC(\mathfrak{g}_1)/\sim \xrightarrow{\simeq} MC(\mathfrak{g}_2)/\sim$

This is why we need an $\hbar$: if it precedes α, for a fixed power of $\hbar$, only some of these terms will contribute.

Proof is an exercise.
Extending U_1:

- first for $M = \mathbb{R}^d$
- globalization (choice of connection)

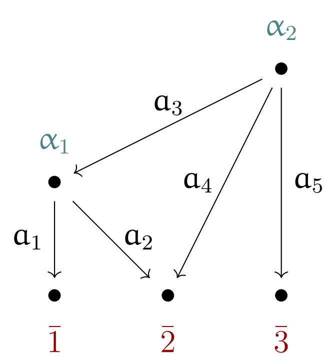
Consider $U_k : \Lambda^k T_{\text{poly}}(\mathbb{R}^d) \to D_{\text{poly}}(\mathbb{R}^d)$. Explicitly,

$$U_k = \sum_{m \geqslant 0} \sum_{\Gamma \in G_{n,m}} W_\Gamma \cdot U_\Gamma$$

where $W_\Gamma \in \mathbb{C}$, $U_\Gamma \in \mathrm{Hom}(\otimes^n T_{\text{poly}}(\mathbb{R}^d), D^{m-1}_{\text{poly}}(\mathbb{R}^d))$, and Γ is an admissible graph with:

- n aerial vertices $1, \dots, n$,
- m ground vertices $\bar{1}, \dots, \bar{m}$,
- $2n + m - 2$ edges, only from aerial vertices with no multiple edges and no loops (they can end at aerial or ground edges)
- and a choice of order of k_i edges emanating from the aerial vertex i.

Now U_Γ takes n polyvectors and evaluates on m functions. Suppose Γ looks like:



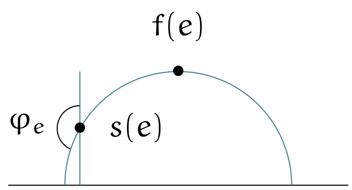
U_Γ is only nonzero if α_1 is a k_i multivector. Then

$$U(\alpha_1, \alpha_2)(f_1, f_2, f_3) = \partial_{a_3}(\alpha_1)^{a_1 a_2}(\alpha_2)^{a_3 a_4 a_5}\, \partial_{a_1} f_1\, \partial_{a_2}\, \partial_{a_4} f_2\, \partial_{a_5} f_3.$$

note that the a_is are indices. Then

$$W_\Gamma = \prod_{i=1}^{n} \frac{1}{(2\pi)^{k_i} k_i!} \int_{f(e)} d\varphi_{e_1} \dots d\varphi_{e_n}$$

where φ is a function on moduli space $\bar{C}_{n,m}$ of all ways to place the vertices on the upper half plane, the aerial vertices above the real axis; the ground vertices on the real axis. These configurations are considered up to $z \to az + b, a > 0, b \in \mathbb{R}$. The angle associated to a vertex e going from vertex $s(e)$ to vertex $t(e)$ is measured as on the picture



The numbers W_Γ are notoriously hard to compute.

Why do we care about this at a TQFT school? We can compute $f \star g(x)$ as a path integral for a specific TQFT constructed using the Poisson manifold M, the so-called Poisson σ-model (PSM)

$$f \star g(x) = \int_{X(\infty)=x} f(X(1))g(X(0))e^{\frac{i}{\hbar}S_{\mathrm{PSM}}(X,\eta)}\mathcal{D}X\mathcal{D}\eta,$$

whose Feynman diagrams are the graphs we just talked about. So we can compute star products as an expectation value of some TQFT. To read more, see [CF00]. Here $X : D_{\mathrm{disk}} \to M$ (from a disk with three points labelled $0, 1, \infty$ on the boundary) with $\eta \in \Gamma(D, X^*(T^*M) \otimes T^*D)$.

q 4.3 **Theo:** is η a fermion?
Ján: Not quite, η are ghosts which satisfy fermionic statistics. *(this was wrong as Theo then pointed out.)* The BV action is of the form

$$S_{\mathrm{PSM}} = \int_D \eta_i \wedge dX^i + \frac{1}{2}\pi^{ij}\eta_i \wedge \eta_j.$$

It is a 2-dimensional theory of AKSZ type.
Theo: In the above action, the fermionic fields have already been integrated out.

SECTION 4.2

Next time

Different deformation quantizations using a different TQFT: Chern–Simons.

- quantization of Lie bialgebras/Poisson–Lie groups (sector of Chern–Simons with some boundary conditions)
- quantization of moduli spaces of flat connections over a surface (phase spaces, which we can get as endomorphisms of some object in the category below)
- quantization of categories (in physics, categories of line observables of Chern–Simons theory)

In all three constructions, we can use Drinfeld associator. Even gives deformation quantization of Poisson manifold

q 4.4 **Sal:** what does quantization mean in math?
Ján: Depending on context, e.g. Lie bialgebras and Poisson algebras, which we use in the first two points. In the third case, vibes are it makes commuting things commute less (symmetric monoidal to braided monoidal). In general, people say quantization is not a functor.
Theo: This was one of my first MathOverflow questions[a]. Theo gave roughly Ján's answer, but there are also really good answers and different perspectives.

[a] https://mathoverflow.net/questions/6200/what-is-quantization/6216

q 4.5 **Luuk:** in the action the Poisson structure is on the target?
Ján: Yes, you don't use geometric structure on the source. You pull back through X.

q 4.6 **Sal:** Why does this have to be in 2D?
Ján: (partial answer) Poisson manifold gives a 1-shifted symplectic manifold $T^*[1](M)$. The space of fields in AKSZ are maps from $T[1]$ of source (in our case D) which is dimension $n+1$ and X has dimension n. $n-(n+1)=-1$ which is the right number to do BV formalism.

Day 5

DAY
5

Today we will get more handwave-y.

SECTION 5.1

Deformation quantization of categories

The general principle of deformation quantization: start with a commutative algebra of classical observables, replace with associative algebra of quantum observables:

	$\hbar^1$	$\hbar^2$	
{commutative algebra of classical observables}	Poisson bracket	Jacobi	{associative algebra of quantum observables}
{symmetric monoidal algebra of line observables}	infinitesimal braiding $t \in (\mathrm{Sym}^2\,\mathfrak{g})^{\mathfrak{g}}$	automatic?	{braided monoidal category of quantum observables}

$\hbar^1$ encodes the Poisson bracket, $\hbar^2$ encodes Jacobi.

There are line observables in classical Chern–Simons, which form a symmetric monoidal category ($U\mathfrak{g}$-modules). They quantize to a braided[2] monoidal category ($U\mathfrak{g}\text{-mod}^{\Phi}_{\hbar}$). $\hbar^1$ encodes infinitesimal braiding $t \in (\mathrm{Sym}^2\,\mathfrak{g})^{\mathfrak{g}}$[3], and $\hbar^2$ in this case is automatic(?).

[2] since codimension of lines in 3-space is 2, and E_2 means braided

[3] i.e. t is an invariant element

def 5.1 Let $\mathfrak{g}$ be a Lie algebra, $t \in \left(\mathrm{Sym}^2\,\mathfrak{g}\right)^{\mathfrak{g}}$, e.g. $\mathfrak{g}$ with a nondegenerate invariant pairing (any semisimple Lie is such, with Killing form). Define $U\mathfrak{g}\text{-mod}^{\Phi}_{\hbar}$ to be the category with:

- Same objects as $U_{\mathfrak{g}}$
- $\mathrm{Hom}_{U\mathfrak{g}\text{-mod}^{\Phi}_{\hbar}}(X, Y) = \mathrm{Hom}_{U\mathfrak{g}\text{-mod}}(X, Y)[\![\hbar]\!]$
- composition, $\otimes$ extended $\hbar$-linearly.

Then, we also need an associator and braiding. For the braiding $X \otimes Y \to Y \otimes X$, we'll use

$$\sigma_{X,Y} \circ e^{\frac{\hbar}{2} t_{X,Y}} \tag{5.1}$$

where $e^{\frac{\hbar}{2} t_{X,Y}} \in (U_{\mathfrak{g}} \otimes U_{\mathfrak{g}})[\![\hbar]\!]$, $t_{X,Y} = \rho_X \otimes \rho_Y(t)$, $\rho_X : \mathfrak{g} \to \mathrm{End}(X)$ and $\sigma_{X,Y}$ is the flip map. Note that $t = t^{ij} e_i \otimes e_j$, and

$$t_{X,Y}(x \otimes y) = t^{ij} \rho_X(e_i)(x) \otimes \rho_Y(e_j)(y)$$

Our associator $\tilde{\Phi} \in (U_{\mathfrak{g}} \otimes U_{\mathfrak{g}} \otimes U_{\mathfrak{g}})[[\hbar]]$ takes

$$(X \otimes Y) \otimes Z \xrightarrow{\tilde{\Phi}_{X,Y,Z}} X \otimes (Y \otimes Z).$$

We'll use

$$\tilde{\Phi} = \Phi(\hbar t \otimes 1, 1 \otimes \hbar t)$$

where $\Phi \in \mathbb{C}\langle\langle X, Y\rangle\rangle$ (formal power series in 2 noncommutative variables) such that

$$\Phi = 1 + \frac{XY - YX}{24} + \cdots$$

satisfying the pentagon and hexagon equations. They are equations Φ has to satisfy in order for $\tilde{\Phi}$ to satisfy pentagon and hexagon equations.

Such $\Phi(X, Y)$ is called a[4] *Drinfeld associator*.

[4] as far as we know, only 3 Drinfeld associators are known explicitly, of which two are "opposites" of each other.

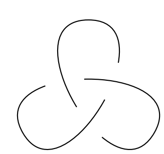
thm 5.2 (Drinfeld)
- The set of Drinfeld assocations is nonempty (it even contains a rational Drinfeld associator)
- Is a torsor for 2 commuting group actions (a bitorsor) which Drinfeld called the *Grothendieck–Teichmüller group* GT and GRT (GRT is the "associated graded" to GT).

q 5.3 **Kyle:** Why do we only know 3 Drinfeld associators?
Ján: One was constructed by Drinfeld, monodromy of the KZ equation. Another from Kontsevich quantization formula. Both have big integrals as coefficients. Also newer work by people like Brown on number theoretic aspects. Now there is more known about rational Drinfeld associator.

q 5.4 **Luuk:** what is a bitorsor?
Ján: A bitorsor is a set with two group actions such that it's a torsor for each and the actions commute

Suppose we have a knot K[5]. and we choose a finite dimensional representation of $\mathfrak{g}$. Then we can understand this knot as a morphism in the category $U_{\mathfrak{g}}\text{-mod}_{\hbar}^{\Phi}$. Then crossings correspond to $e^{\frac{\hbar}{2} t_{X,X}}$ and cups some normalization of coev, etc. Note that in this way, $K_X \in \mathrm{Hom}_{U_{\mathfrak{g}}\text{-mod}_{\hbar}^{\Phi}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}[[\hbar]]$.

[5] e.g.

K = 

In Chern–Simons theory,

$$K_X = \langle \mathrm{Wilson}_{K,X} \rangle_{CS} = \int \mathcal{D}A e^{S_{CS}(A)/\hbar} \operatorname{tr}_X \mathrm{hol}_K(A).$$

SECTION 5.2

Deformation quantization of Lie bialgebras

def 5.5 A *Lie bialgebra* is a Lie algebra $\mathfrak{g}$ with a map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ such that:

- $\mathfrak{g}^*$, δ^* is a Lie algebra, and
- $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is a Lie algebra cocycle.

A deformation quatnization of $\mathfrak{g}$ is a deformation of $U_\mathfrak{g}$ as a Hopf algebra such that

$$\lim_{\hbar \to 0} \frac{\Delta - \sigma\Delta}{\hbar} = \delta$$

ex 5.6 For $\mathfrak{g}$ semisimple (q is even *quasi-triangular*: $\delta = d_{CE}(r)$, $r \in \mathfrak{g} \otimes \mathfrak{g}$ special)

thm 5.7 (Drinfeld–Jimbo) There is a Hopf algebra $U_q\mathfrak{g}$ for $\mathfrak{g}$ semisimple which is a deformation quantization of the Lie bialgebra structure. Moreover, there is an element $R \in U_q\mathfrak{g} \hat{\otimes} U_q\mathfrak{g}$ such that in the category of $U_q\mathfrak{g}$-modules we have a braiding

$$X \otimes Y \xrightarrow{R} X \otimes Y \xrightarrow{\sigma} Y \otimes X.$$

Modules over Hopf algebra is a category which is *monoidal* (from the coproduct).

thm 5.8 (Drinfeld, Kohno) There is an equivalence of braided monoidal categories

$$U_q\mathfrak{g}\text{-mod} \xrightarrow{\cong} U_\mathfrak{g}\text{-mod}_\hbar^\Phi$$

Lie bialgebras in general can be quantized ([Etingof–Kazhdan]). Another way to quantize them comes from TQFT. Let $(\mathfrak{g}, [,], \delta)$ be a Lie algebra, and consider a Chern–Simons theory for $\mathfrak{g} \oplus \mathfrak{g}^*$. Construct a functor:

$$N : \triangle \subset \mathrm{FinSet} \longrightarrow \mathrm{Vect}$$

where we consider the simplex category $\triangle$ as a wide (has all objects but only some morphisms) subcategory. We take $N([n]) = Z^{CS}_{\text{classical}}(\mathfrak{g} \oplus \mathfrak{g}^*)(M)$ where M is a surface isomorphic to (n)-pants. The boundary condition at the waist is given by $\mathfrak{g}$ and the boundary condition in each pant leg by $\mathfrak{g}^*$.

thm 5.9 Symmetric lax monoidal functors

$$\mathrm{FinSet} \to \mathrm{Vect}$$

can be used to faithfully encode commutative Hopf algebras, such as (the dual of) $U\mathfrak{g}$.
To describe Poisson brackets on Hopf algebras (e.g. Lie cobrackets on $\mathfrak{g}$), replace finite sets with infinitesimally braided sets; to describe general Hopf algebras, replace finite sets with *braided* sets:

$$\begin{array}{ccc} [\mathrm{iBraidedSet}, \mathrm{Vect}] & \longleftarrow & \text{Poisson–Hopf algs } (U_{\mathfrak{g}}, \delta)^* \\ \downarrow{\scriptstyle \circ\Phi} & & \\ [\mathrm{BraidedSet}, \mathrm{Vect}] & \longleftarrow & \text{Hopf algs } (U_q\mathfrak{g})^* \end{array} \tag{5.2}$$

so we can quantize Lie bialgebras by precomposition with the Drinfeld associator.

SECTION 5.3

Deformation quantization of moduli spaces of flat connections

For moduli spaces of flat connections, we can deformation quantize using Skein theory: skeins on surface describe functions on the moduli space. There is a nice non-commutative algebra given by skeins in the surface times the interval; choosing a section requires some combinatorial work.

SECTION 5.4

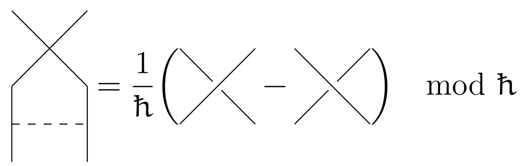
Final remarks

We've talked about deformation quantization (about observables). There is also geometric quantization: you need to choose half of coordinates in some way (choice of polarization) since phase space is (p, q) and we end up only with q. Geometric quantization tries to do more, so it's harder to prove things about it. There is a modern perspective on deformation quantization called Brane quantization. It's a string-theory inspired way to get both DQ and GQ at once. After this course, the introduction to [GW09] is rather readable.

q 5.10 How do quantum groups relate to groups?
Ján: Taking a group, if Lie algebra is Lie bialgebra, you have a Poisson structure on the group. You can deformation quantize and get functions on G: $C^\infty(G)$ with star product. Additionally the quantum group quantizes the coproduct (coming from the group multiplication) so that quantum groups are Hopf algebras (regular groups are (co)commutative Hopf algebras).

q 5.11 **Luuk:** What is a braided set? Isotopy classes of braids?
Ján: Yes but strands can collide (just like how maps of sets can map two elements to the same element).
Luuk: So it's a map of sets where you remember the over and under.

q 5.12 What is infinitesimal braided?
Ján: It's a map of sets (ignoring over and under) with cords that satisfy a Drinfeld-Kohno relations



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[GW09] Sergei Gukov and Edward Witten. Branes and Quantization. *Advances in Theoretical and Mathematical Physics*, 13(5):1445–1518, 2009.

Deformation Quantization
References and further reading

Ján Pulmann

June 2025

The definition of star products is due to [BFFLS78]. These authors introduced the concept of deformation quantization to capture the algebra of quantum observables, and noticed it makes sense for general Poisson manifolds.

Regarding books and lecture notes, I particularly enjoyed the notes by [Gut05], the book of Fedosov [Fed96], first parts of Kontsevich's paper [Kon03][1] and the notes by Weinstein [Wei94]. Below follow references for individual lectures.

Lecture 1

The history of quantum mechanics in phase space, including Dirac's reluctance and multiple independent discoveries, are nicely presented in the historical overview [CZ12]. Their book [ZFC05] contains more details and many important papers. The calculation of the Moyal product from the Weyl-Wigner transition is taken from [Fed96, Ch. 3]. For those interested in functional-analytic treatment, see the book by Folland [Fol89].

Lecture 2

The "alternative" definition of deformation quantization can be found in [Del95] and [RS02, Sec. 3.2].

The algebraic structure $\circ$, $[-,-]$ on Hochschild cochains can be found in Gerstenhaber's original paper [Ger63], see also [Kon03] for a concise overview.

See e.g. [Sch23] for a detailed treatment of $U\mathfrak{g}$ and the induced star product on $\mathfrak{g}^*$. Rees construction and the relationship between filtered and formal deformation quantizations is explained in [BRSSW16, Sec. II.2.6]. The example of the quantum torus appears in e.g. [RS02].

Lecture 3

See e.g. [Mei18, Sec. 1] for a short introduction into Poisson geometry; there are many books on symplectic and Poisson manifolds.

A nice overview of Fedosov's result and the classification of star products on symplectic manifolds is in [Gut05]. See also Fedosov's book [Fed96].

For Poisson structures on moduli spaces of flat connections, see [AB83; FR99].

The content on HKR isomorphism is from [Kon03], see also [DMZ07] and [DSV23] specifically for Maurer-Cartan elements.

Lecture 4

In addition to [Kon03], the explicit formula for Kontsevich's L_∞ quasi-isomorphism is also explained well in [Gut05], which also uses the terminology of "aerial" and "ground" vertices. The relationship with the Poisson σ-model is due to Cattaneo and Felder [CF00].

[1] Particularly, see the version on Kontsevich's website linked in the reference.

Lecture 5 (+ end of lecture 4)

Drinfeld associators were introduced by Drinfeld in [Dri90]. See also [Car93] for relationships with braided monoidal categories, as well as Chapter XIX of Kassel's book [Kas95]. We used the deformation quantization perspective on Drinfeld categories in [KKMP24, Sec. 5]. For Poisson-Lie groups and Lie bialgebras, see Drinfeld's [Dri88], as well as Kassel's book [Kas95]. Lie bialgebras were first quantized by [EK96], the quantization I sketched is from [PŠ22]. Finally, the quantization of moduli spaces of flat connections via categorical deformations is due to [BBJ18], see [KKMP24] for the case of the Drinfeld category.

Other References

I also mentioned brane quantization of Gukov and Witten [GW09]. For geometric quantization (as well as its shifted upgrade), I recommend the introduction to [Saf23] and references therein.

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