DIAGRAMMATIC CALCULUS AND SKEIN THEORY

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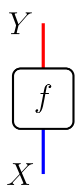
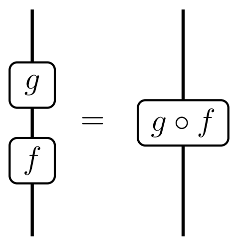
1. Diagrammatic calculus and monoidal categories

Diagrammatic calculus is a useful way for understanding monoidal categories in a visual format. By convention, we will read all diagrams from bottom to top. For now, lines in a diagram are objects, coupons (boxes) are morphisms, dots are morphism spaces, and dotted lines are the unit object. We will update this notation later as there is a more convenient way of drawing things. In what follows, we fix some monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$. We will use colored labels when it helps clarify things, but we won't always use different colors for different objects.

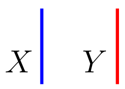
We represent the morphism $f \in \mathcal{C}(X, Y)$ by the diagram



When it is obvious from context, we will omit the object labels and object colors. For morphisms $f \in \mathcal{C}(X, Y), g \in \mathcal{C}(Y, Z)$ we have



The tensor product $X \otimes Y$ is drawn as



For a morphism $f \in \mathcal{C}(X \otimes Y \to Z)$, we draw

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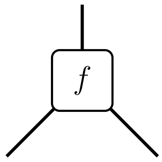
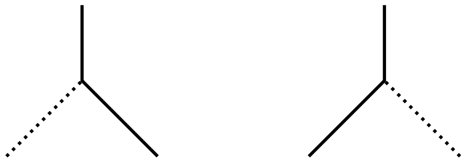
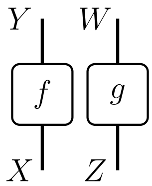


We won't draw coupons for unitors. The left and right unitors λ_X, ρ_X are shown below on the left and right, respectively.

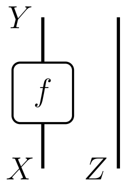


A diagram which looks like an upside down unitor junction is the inverse of that unitor.

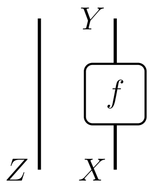
For $f \in \mathcal{C}(X,Y), g \in \mathcal{C}(Z,W)$, the morphism $f \otimes g \in \mathcal{C}(X \otimes Z, Y \otimes W)$ is written as



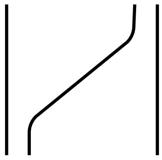
We won't draw coupons for identity morphisms. So for $f \in \mathcal{C}(X,Y)$, we write $f \otimes 1_Z$ as



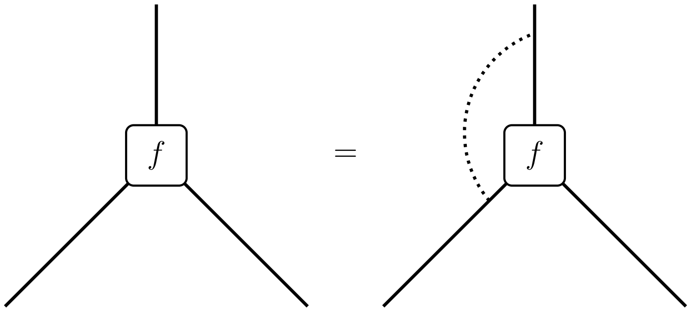
and $1_Z \otimes f$ as



If we draw three lines next to each other, that represents the product of some objects X, Y, Z. However, this object depends on parenthesization. Our convention will be that objects which are drawn closer are parenthesized together. This introduces some level of ambiguity if we are not careful, but by giving sufficient context and drawing the diagrams carefully this will not be an issue. This is consistent with how we have have drawn morphisms $\mathcal{C}(X \otimes Y, Z)$ previously. We represent the associator $\alpha_{X,Y,Z}$ (without a coupon) by



One benefit of drawing things this way is that we may omit the dotted line for the identity. The coherence conditions on monoidal categories guarantee that we may attach dotted lines which begin and end in a diagram without changing the morphism. For example, for $f \in \mathcal{C}(X \otimes Y, Z)$



Algebraically this is the identity $f = \lambda_Z \circ (1_I \otimes f) \circ \alpha_{I,X,Y} \circ (\lambda_X^{-1} \otimes 1_Y)$.

References