FUSION CATEGORIES AND PHYSICS: PROBLEM SET 1

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1. Tannaka-Krein for groups

Let $F : \mathsf{Rep}G \to \mathsf{Vec}$ be the forgetful functor from the category of finite dimensional representations of a finite group G to the category of finite dimensional vector spaces. Prove that $\mathrm{Nat}(F, F)$ is a group and is isomorphic to G.

2. Algebras

Let $\mathcal{C}$ be a $\mathbb{C}$-linear category and let M be an object in $\mathcal{C}$. Prove that $End(M)$ is an algebra.

Now suppose that $\mathcal{C}$ is also monoidal with a linear monoidal product $\otimes$. Suppose now that $M \in \mathcal{C}$ and there is some $m : M \otimes M \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[rr] \arrow[d, "m\otimes 1_M"] & & M \otimes (M \otimes M) \arrow[d, "1_M \otimes m"] \\
M \otimes M \arrow[dr, "m"] & & M \otimes M \arrow[dl, "m"] \\
& M &
\end{tikzcd}$$

Prove that $End(M)$ is a commutative algebra.

3. Bialgebras

Let A be a finite dimensional bialgebra with coproduct Δ. Prove the following:

(1) If $\pi : A \to B(V)$ and $\rho : A \to B(W)$ are finite dimensional A-modules, then $\pi \bigotimes \rho := (\pi \otimes \rho) \circ \Delta$ is also a finite dimensional representation.
(2) $\pi \bigotimes (\rho \bigotimes \eta) \cong (\pi \bigotimes \rho) \bigotimes \eta$. Call this associator $\alpha_{\pi,\rho,\eta}$.
(3) Prove that α obeys the pentagon equation.
(4) Prove that ${}_A\mathrm{Mod}$ is a semisimple monoidal category.

If you prefer, you may do this all with right modules.

4. Classification of monoidal structuress

Let $\mathsf{Vec}G$ be the category of G-graded finite dimensional vector spaces. On objects, the $g \in G$ component of the tensor product of G-graded spaces $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$ is defined by

$$(V \otimes W)_g := \bigoplus_{h\in G} V_{gh} \otimes W_{h^{-1}}.$$

Date: May 19, 2025.

Classify, up to linear monoidal equivalence, all monoidal categories whose underlying $\mathbb{C}$-linear category is $\mathsf{Vec}G$. Even if you know the answer, you should prove this using the basic definition of monoidal categories and their functors. *Hint: Use group cohomolgy. If you don't know group cohomology, you may simplify to the case where* $G = \mathbb{Z}/2\mathbb{Z}$*. For simplicity, you may also use the skeletal version of* $\mathsf{Vec}G$*.*