FUSION CATEGORIES AND PHYSICS: PROBLEM SET 2

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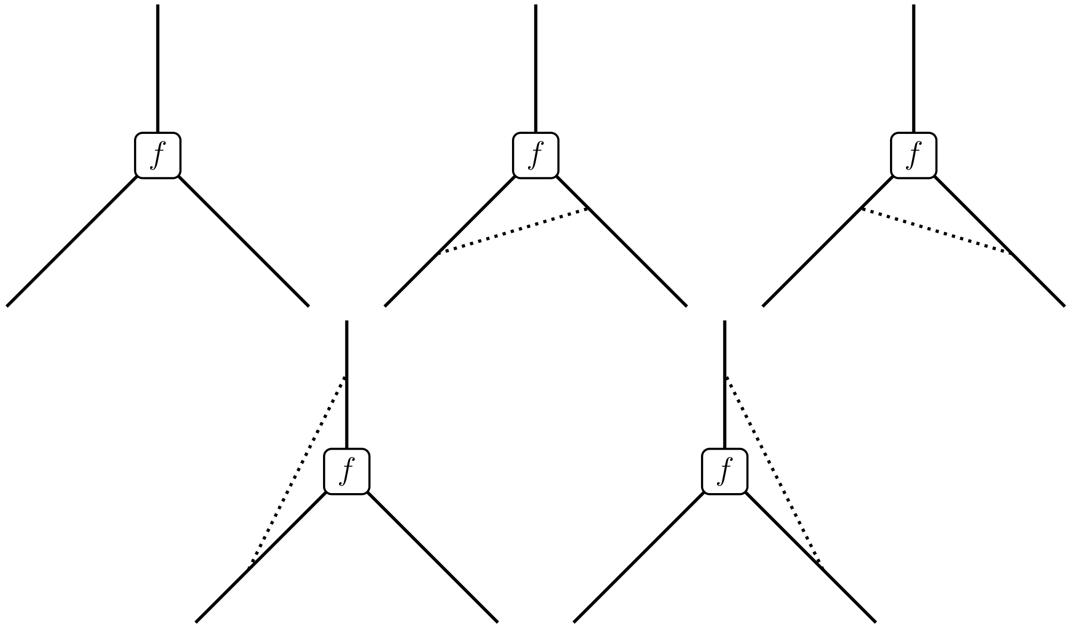
1. Unit diagrams

Let $\mathcal{C}$ be a monoidal category with unit object I, let $X, Y, Z \in \mathcal{C}$, and let $f \in \mathcal{C}(X \otimes Y \to Z)$. Translate the following diagrams into compositions of morphisms and prove they are all equal.



Hint: Don't forget about naturality!

2. Idempotent complete categories

Let $\mathcal{C}$ be idempotent complete. Prove that an object in $\mathcal{C}$ is simple if and only if it is indecomposable.

Note: Be careful to not just prove a tautology!

3. Double duals

Let $\mathcal{C}$ be a multifusion category and let $X \in \mathcal{C}$. Prove that $X^\vee \cong {}^\vee X$. Deduce that $(X^\vee)^\vee \cong X$.

4. Coherence conditions

Using the diagrammatic calculus draw the coherence conditions for monoidal categories, monoidal functors, and monoidal natural transformations.

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5. Fusion dimensions

Let $\mathcal{C}$ be a fusion category and define $N_{X,Y}^{Z}$ for simple objects $X, Y, Z \in \mathcal{C}$. Prove that for simple $A, B, C, D \in \mathcal{C}$

$$\sum_{Z \in Irr(\mathcal{C})} N_{Z,C}^{D} N_{A,B}^{Z} = \sum_{Z \in Irr(\mathcal{C})} N_{A,Z}^{D} N_{B,C}^{Z}$$

where the sum is taken over (isomorphism classes of) simple objects.

6. Frobenius-Perron dimension

Let $\mathcal{C}$ be a fusion category. For an object $X \in \mathcal{C}$ whose simple decomposition is $X \cong \bigoplus_{r \in Irr(\mathcal{C})} N_r r$ where the sum is taken over (isomorphism classes of) simple objects in $\mathcal{C}$ and we are taking N_r direct sums of object r in this sum, define the *object norm* (this is a non-standard term) as

$$|X| := \sqrt{\sum_{r \in Irr(\mathcal{C})} N_r^2}.$$

Use the Frobenius-Perron theorem to prove that for all $X \in \mathcal{C}$, there exists some $\lambda_X > 0$ such that for all $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \lambda_X < \beta$,

$$\lim_{m \to \infty} \frac{|X^{\otimes m}|}{\beta^m} = \lim_{m \to \infty} \frac{\alpha^m}{|X^{\otimes m}|} = 0.$$

This number λ_X is called the *Frobenius-Perron dimension* of X.