FUSION CATEGORIES AND PHYSICS: PROBLEM SET 3

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1. Group doubles

Let G be a finite group. The quantum double algebra $D[G]$ is given by the complex span of elements (g,h) for $g,h \in G$ and has multiplication generated by

$$(g,h)(k,l) = \delta_{h,lkl^{-1}}(gk,l).$$

Prove that $D[G] \cong Tube(\mathsf{Vec}G)$.

(Note: The rest of this exercise is significantly harder.) Find a coproduct (without looking it up) and counit which turns $D[G]$ into a bialgebra. Prove that this is a bialgebra. Can you think of a skein theoretic interpretation of the coproduct? Hint: Defining the idempotent $P := \frac{1}{|G|}\sum_{g\in G}(g,1)$, we have $PD[G]P \cong \mathbb{C}[G]$. You may find it helpful to note that bialgebras are designed to be the types of algebras whose modules you can take tensors of. Recall that representations of groups also have this property.

In the case where G is abelian, find explicit forms for each irrep of $D[G]$ and find their fusion rules under the tensor product.

2. Spontaneously broken translation symmetry

(Note: Mathematicians may want to pair up with their physicist friends for help with the intuition here.)

Consider the following properties of a length $3N$ qudit chain Hilbert space:

(1) The Hilbert space is $\mathcal{H} = \otimes_{k=1}^{3N}\mathbb{C}^d$ (for some fixed d of your choice) with an orthonormal basis labeled by $\{|1\rangle, ..., |d\rangle\}$ on each site. Take the basis of $\mathcal{H}$ to be labeled strings of integers in $[1,d]$ written as $|s_1,...,s_{3N}\rangle$.
(2) There is an operator $T : \mathcal{H} \to \mathcal{H}$ which acts as $T|s_1,s_2,...,s_{3N}\rangle = |s_{3N},s_1,s_2,...,s_{3N-1}\rangle$.

Invent a family of quantum qudit chains on this ring of length $3N$ with the following properties:

(1) The Hamiltonian is of the form $H = \sum_{k=1}^{3N} H_k$ where H_2 is an operator supported on the first 3 tensor factors of $\mathbb{C}^d$ in $\mathcal{H}$. We also impose that $H_{k+1} = TH_kT^\dagger$ (where we take $N+1 = 1$ in the index). This implies that $[H,T] = 0$.
(2) The ground state subspace is 3 dimensional.
(3) The ground state subspace has at most 1 dimension where T has eigenvalue 1.

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3. Drinfeld center

Let G be an abelian group. Find all simple objects of $Z(\mathsf{Vec}G)$ and find all structure isomorphisms on these objects for $Z(\mathsf{Vec}G)$ as a monoidal category. Also find the braiding isomorphisms.

4. Tube algebra as C^*-algebra

(Note: This is the most important exercise.)

Let $\mathcal{C}$ be a unitary fusion category. Prove that $Tube(\mathcal{C})$ is a finite dimensional C^*-algebra where the $*$ is induced from the dagger structure on $\mathcal{C}$.