FUSION CATEGORIES AND PHYSICS: PROBLEM SET 4

KYLE KAWAGOE

1. Ground state

How many ground states can you find for the Levin-Wen model when defined on the sphere? What if it's defined on the torus?

Note: It may be too difficult to prove that you've found them all. This is why the problem is phrased as it is.

2. Quadratic forms

Let G be an abelian group and let $q : G \to \mathbb{C}^\times$ be a quadratic form. That is, $q(g) = q(g^{-1})$ and $b : G \times G \to \mathbb{C}^\times$ is a homomoprhism in both arguments where $b(g,h) = \dfrac{q(gh)}{q(g)q(h)}$. A non-degenerate quadratic form is one such that as a $|G| \times |G|$ matrix, $det(b) \neq 0$. Prove that if q is non-degenerate, there exists a braided category $\mathsf{Vec}^q G$ which is $\mathsf{Vec}G$ as a monoidal category, but has a braiding c such that

$$c_{g,g} = q(g)1_g$$

and

$$c_{h,g} \circ c_{g,h} = b(g,h)1_{gh}.$$

Note that we are taking $\mathsf{Vec}G$ to be skeletal.

3. String operator braiding

Consider the Levin-Wen model for a unitary fusion category $\mathcal{C}$. Let $(X,\psi) \in Z(\mathcal{C})$ and let $S^{(X,\psi)}_{\alpha,\beta}$ be the string operator which terminates using the morphisms $\alpha \in \mathcal{C}(\Gamma, X)$ and $\beta \in \mathcal{C}(X, \Gamma)$ where Γ is the sum of all simples in $\mathcal{C}$. Now let $(Y,\phi) \in Z(\mathcal{C})$. Suppose that $L^{(Y,\phi)}$ a closed loop string operator which encloses the α end point of $S^{(X,\psi)}_{\alpha,\beta}$. Then there is some linear map $M : \mathcal{C}(\Gamma, X) \to \mathcal{C}(\Gamma, X)$ such that

$$L^{(Y,\phi)} S^{(X,\psi)}_{\alpha,\beta} = S^{(X,\psi)}_{M(\alpha),\beta} L^{(Y,\phi)}.$$

Describe M in terms of $\beta, (X,\psi), (Y,\phi)$ and the categories $Z(\mathcal{C})$ and $\mathcal{C}$.

Make sure S and L are large so that the end points of the string are far enough away from the loop for this question to make sense!

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4. Tube algebra representation

Let $\mathcal{C}$ be a unitary fusion category and let Γ be the sum of all simples in $\mathcal{C}$. For simple $(X, \psi) \in Z(\mathcal{C})$, come up with a faithful representation of $Tube(\mathcal{C})$ on the space $\mathcal{C}(X, \Gamma)$. Note that you will need to sues ψ. Prove that if $(X, \psi), (Y, \phi) \in Z(\mathcal{C})$ are distinct simples, then the representation induced by (X, ψ) and (Y, ϕ) are not isomorphic as representations. This is the first step in showing that ${}_{Tube(\mathcal{C})}Mod \cong Z(\mathcal{C})$.

5. Physics translation

Let $\mathcal{C}$ be a unitary fusion cateogry of your choice. Let $\mathcal{H}_v$ be the Levin-Wen Hilbert space for $\mathcal{C}$ on site v. Also let $S_v \subset B(\mathcal{H}_v)$ (this just means operators from $\mathcal{H}_v$ to itself) be a set of operators of your choice. Express the Hamiltonian of the Levin-Wen model as a polynomial of these operators. By polynomial, we mean that you can take both tensor products and do operator composition. For example, $(A \otimes B)E + 6C \otimes (CD)$ is a polynomial of the operators A, B, C, D, E.