Fusion Categories and Condensed Matter

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Contents

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A big thanks to Luuk and Theo for organizing the conference, to the University of New Brunswick and Branimir for hosting us, to the instructors for the care they took in preparing lectures and writing exercises, to the TAs for their generosity and thoughtfulness, and to the fellow students for cultivating a community.

The land on which UNB is situated is the unceded and unsurrendered territory of the Wolastoqey (Maliseet)[1], whose territories are governed by the Peace and Friendship Treaties[2], which recognized Wolastoqey title, and established rules for what was to be an ongoing relationship with the Nations.

Disclaimer (inspired by Theo's[3]): As is the nature of notes, the information Kyle communicated in lecture has only been encoded in these notes after being processed in the black box that is my brain. Anything you find helpful or creative or insightful is due to Kyle, and anything you find misleading or incorrect is due to me. Please let me know[4] if you find errors/corrections!

[1] `https://wnnb.wolastoqey.ca/about-us/our-history/`

[2] `https://www.rcaanc-cirnac.gc.ca/eng/1100100028589/1539608999656`

[3] `https://categorified.net/otherdocs.html`

[4] `tiand@umich.edu`

DAY 1

Day 1

Prereqs: you should know what a category is and what a finite-dimensional Hilbert space is.

SECTION 1.1
Motivation

Your space of "how things are" (states) is a *state space*, usually Hilbert spaces. If a Hilbert space has symmetry transformations, they naturally come about from representations of groups. We want our categories to have the complex numbers and a way combine things, which leads us to *fusion categories*.

def 1.1 In gory detail, a *monoidal category* is a 6-tuple (which we'll mostly only think about part of) $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ where:

- $\mathcal{C}$ is a category
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a functor
- the *unit* $I \in \mathcal{C}$ a special object (in the future these will be written 1)
- the *associator* $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$ a natural isomorphism
- the *left unitor* $\lambda : I \otimes - \to -$ a natural isomorphism where $-$ is the identity functor on $\mathcal{C}$
- the *right unitor* $\lambda : - \otimes I \to -$ a natural isomorphism

subject to the following coherence conditions (i.e. they are related to each other in a way that makes sense):

- The pentagon diagram: Let $A, B, C, D \in \mathcal{C}$, and as shorthand we write $A \otimes B =: AB$

$$\begin{array}{ccccc} & & (A(BC))D & \xrightarrow{\alpha_{A,BC,D}} & A((BC)D) \\ & \nearrow^{\alpha_{A,B,C}\otimes 1_D} & & & \downarrow 1_A\otimes\alpha_{B,C,D} \\ ((AB)C)D & & & & \\ & \searrow_{\alpha_{AB,C,D}} & & & \\ & & (AB)(CD) & \xrightarrow{\alpha_{A,B,CD}} & A(B(CD)) \end{array} \tag{1.1}$$

This can also be written in terms of tree diagrams.

- The triangle diagram:

$$\begin{array}{ccc} (XI)Y & \xrightarrow{\alpha_{X,I,Y}} & X(IY) \\ & {}_{\rho_X\otimes I_Y}\searrow \quad \swarrow_{I_X\otimes\lambda_Y} & \\ & XY & \end{array} \tag{1.2}$$

ex 1.2 The category of (finite-dimensional complex) vector spaces $\mathbf{Vec}$.

ex 1.3 The category $\mathrm{End}(\mathcal{C})$ of endomorphisms of a category $\mathcal{C}$.
We can think of objects of a monoidal category as morphisms of some other category (composition gives some monoidal structure). In this case, we get a *strict* monoidal category: the defining natural isomorphisms are equalities. Sometimes, though a monoidal category *can* be strict, doesn't mean we want to choose them to be.

def 1.4 A *monoidal functor*:

$$(F, \varphi, \iota) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \longrightarrow (\mathcal{D}, \bullet, J, \omega, \ell, r) \tag{1.3}$$

such that

- $F : \mathcal{C} \to \mathcal{D}$ is a functor
- a natural isomorphism $\varphi : F - \bullet F- \to F(- \otimes -)$, i.e. the functor preserves the product (where $F-$ is shorthand for $F(-)$)
- a morphism $\iota : J \to F(I)$.

These satisfy a condition that tells us that the associators work together: for $X, Y, Z \in \mathcal{C}$

- the hexagon diagram commutes

$$\begin{array}{ccc} (FX \bullet FY) \bullet FZ & \xrightarrow{\omega_{FX,FY,FZ}} & FX(FY \bullet FZ) \\ \downarrow \varphi_{X,Y} \bullet 1_{FZ} & & \downarrow 1_{FX} \bullet \varphi_{Y,Z} \\ F(X \otimes Y) \bullet FZ & & FX \bullet F(Y \otimes Z) \\ \downarrow \varphi_{X \otimes Y, Z} & & \downarrow \varphi_{X, Y \otimes Z} \\ F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha_{X,Y,Z})} & F(X \otimes (Y \otimes Z)) \end{array} \tag{1.4}$$

so this tells us how to relate α and ω through φ.

- and for the unitors:

$$\begin{array}{ccc} FX \bullet J & \xrightarrow{1_{FX} \bullet \iota} & FX \bullet FI \\ \downarrow r_{FX} & & \downarrow \varphi_{X,I} \\ FX & \xleftarrow[F(\rho)]{} & F(X \otimes I) \end{array} \qquad \begin{array}{ccc} J \bullet FX & \xrightarrow{\iota \bullet 1_X} & FI \bullet FX \\ \downarrow \ell_{FX} & & \downarrow \varphi_{I,X} \\ FX & \xleftarrow[F(\lambda_X)]{} & F(I \otimes X) \end{array} \tag{1.5}$$

def 1.5 An equivalence of monoidal categories is a pair of monoidal functors such that the underlying functors between categories is an equivalence.

q 1.6 **Luuk:** Why is this the right definition?
Theo: The theorem is that an equivalence in the 2-category of monoidal categories is the same as a 1-morphism in the 2-category of monoidal categories which induces an equivalence in the 2-category of all categories.

def 1.7 Let (F, φ, ι) and (G, θ, η) be two functors between monoidal categories $\mathcal{C}$ and $\mathcal{D}$. A *monoidal natural transformation* is a natural transformation $\gamma : F \Rightarrow G$ such that the following diagrams commutes

- For the tensor structure

$$\begin{array}{ccc} FX \bullet FY & \xrightarrow{\gamma_X \bullet \gamma_Y} & GX \bullet GY \\ \downarrow \varphi_{X,Y} & & \downarrow \theta_{X,Y} \\ F(X \otimes Y) & \xrightarrow{\gamma_{X \otimes Y}} & G(X \otimes Y) \end{array} \tag{1.6}$$

This tells us how to work with the underlying tensor structures.

- for the unitors

$$\begin{array}{ccc} & J & \\ {\scriptstyle\iota}\swarrow & & \searrow{\scriptstyle\eta} \\ FI & \xrightarrow{\gamma_{FI}} & GI \end{array} \tag{1.7}$$

SECTION 1.2

Duals

ex 1.8 Every representation has a dual representation which is close to an inverse. For example, the irreducible representations of $\mathbb{Z}/3$ are $\chi : \mathbb{Z}/3\mathbb{Z} \to \mathbb{C}^\times$ with:

$$\chi_1(a) = e^{4\pi i/3} \qquad \chi_2(a) = e^{-4\pi i/3} \qquad \chi_3(a) = 1. \tag{1.8}$$

So in this case representations are invertible and the dual is the inverse.
There's a fact from representation theory that tells us that the tensor between a representation and a dual gives:

$$(\rho, V) \otimes (\rho, V)^\vee \cong 1 \oplus \cdots \tag{1.9}$$

This is also true in some monoidal categories with the following notion of dual.

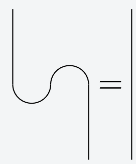
def 1.9 Let $\mathcal{C}$ be a monoidal category with $X \in \mathcal{C}$. We call $X^\vee$ a *right dual* to X if there are morphisms the *evaluation map* and *coevaluation map* with:

$$\mathrm{ev}_X \in \mathcal{C}(X^\vee \otimes X \to I), \qquad \mathrm{coev}_X \in \mathcal{C}(I \to X \otimes X^\vee) \tag{1.10}$$

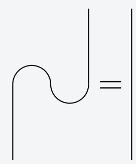
such that:

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \downarrow & & \uparrow \\ I \otimes X & & X \otimes I \\ {\scriptstyle \mathrm{coev}_X \otimes 1_X}\downarrow & & \uparrow{\scriptstyle 1_X \otimes \mathrm{ev}_X} \\ (X \otimes X^\vee) \otimes X & \longrightarrow & X \otimes (X^\vee \otimes X) \end{array} \tag{1.11}$$

(unlabelled morphisms are the ones from $\mathcal{C}$ being a monoidal category). Note that we can also write this in terms of a *string diagram*:



We require a similar string diagram for $X^\vee$:



ex 1.10 Suppose we have a (finite-dimensional) Hilbert space $\mathcal{H}$ and its dual $\mathcal{H}^\vee$. Then, for some $f : \mathcal{H} \to \mathbb{C}$, we have:

$$\mathrm{ev}_{\mathcal{H}}(f \otimes v) = f(v) = \langle f|v\rangle \tag{1.12}$$

Similarly, for $z \in \mathbb{C}$:

$$\mathrm{coev}_{\mathcal{H}}(z) = z \sum_i |e_i\rangle\langle e_i| \tag{1.13}$$

for e_i an orthonormal basis.

Preview: a *linear category* has the morphisms in $\mathcal{C}$ between objects $\mathcal{C}(X, Y)$ is a (finite-dimensional) vector space. We can take direct sums in linear categories, and objects which are not direct sums of other objects are *simple objects* and in fusion categories, there is a finite list which describes the whole category!

q 1.11 **Sal:** There are also left duals!
Kyle: It's in the notes—soon we'll make left and right duals the same.

q 1.12 **Luuk:** does $\mathrm{End}(\mathcal{C})$ have duals?
Kyle: Probably not.

q 1.13 **Kabir:** is there a construction of a free linear category given a category?
Kyle: Sort of—homs are vector spaces but you need extra stuff to tell you how to sum.
Luuk: I guess you can where the vector space has as its basis the previous set of homs

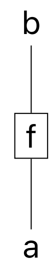
SECTION 1.3

Video on fusion categories

The following notes are based on this video[5].

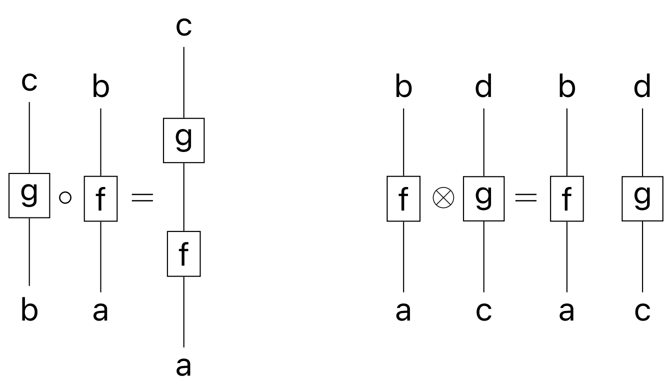
Objects and morphisms in monoidal categories are represented by strings and boxes, and diagrams are read from bottom to top.

$$f : a \to b$$



(1.14)

Composition is given by vertical stacking while the tensor product is given by horizontal juxtaposition:

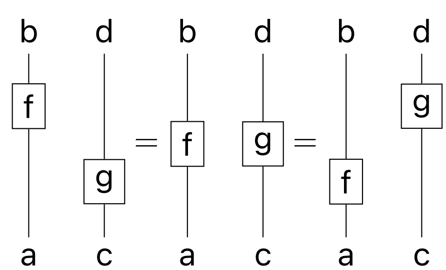


(1.15)

(1.16)

Since the tensor product is associative up to natural isomorphism, we'll suppress parenthization. Similarly, the tensor unit corresponds to a string which can be suppressed.

Additionally, *exchange relations* tell us about compatibility between composition and the product:



(1.17)

[5] https://people.math.osu.edu/penneys.2/Synoptic.mp4

An object a in a monoidal category is *dualizable* if it has a dual $(a^\vee, \mathrm{ev}_a : a^\vee \otimes a \to 1, \mathrm{coev}_a : 1 \to a \otimes a^\vee)$ and predual[1]. A monoidal category is called *rigid* if every object is dualizable.

If a is dualizable, the string for a is oriented from bottom to top (and its dual is oriented top to bottom). For the rest of the definitions, see the original video[6].

[1] I think we're using the terms right- and left-duals

DAY 2

Day 2

Today is adjective day! We'll learn what a fusion category is, and do some diagrammatic calculus.

SECTION 2.1

Fusion categories

We'll begin with a definition which we don't yet know the words for.

def 2.1 A *(multi-) fusion category* is a *linear* monoidal category $\mathcal{C}$ which is

- finitely *semisimple*
- *rigid*
- *idempotent* complete

This category is *fusion* if the unit object $\mathrm{End}(I) \cong \mathbb{C}$.

fusion categories are multi-fusion, but not necessarily the other way around

SUBSECTION 2.1.1

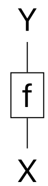
Diagrams

Before we define the things, we'll do some diagrammatic calculus:

- objects are given by lines labeled by $X \in \mathcal{C}$.

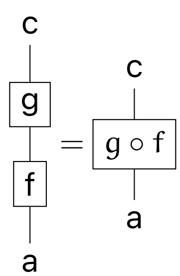
[6] https://people.math.osu.edu/penneys.2/Synoptic.mp4

- morphisms $f \in \mathcal{C}(X \to Y)$ live in a box (are not represented by lines)



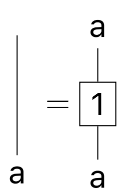
(2.1)

These are read from bottom to top (sources and targets). Things can be composed.



(2.2)

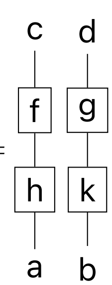
If I draw a line and I tell you it's a morphism, it's the identity morphism



(2.3)

$X \otimes Y$ are given by parallel lines, and $f \otimes g$ is given by parallel morphisms (or a $f \otimes g$ box with two inputs and two outputs).

Since the tensor product is functorial, we have an identity

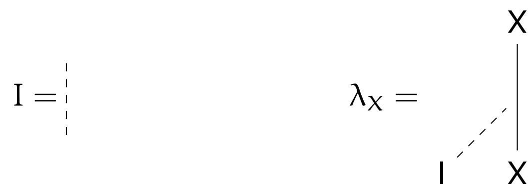
$$(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k) =$$ 

(2.4)

so diagramatic calculus simplifies ambiguities into equalities.

When we write $(X \otimes Y) \otimes Z$, we write the X and Y lines closer together.

(2.5)

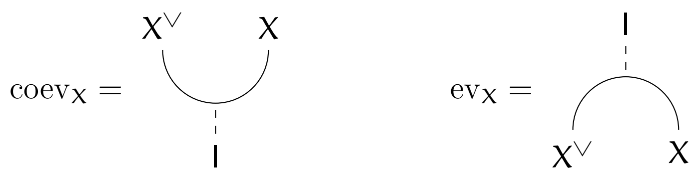


(2.6)

The associator is given by a snakey line in the middle. Whenever objects get regrouped there is an implicit α being applied. Upside-down unitor/associator maps are the inverses.

To prove that diagrammatic calculus is well-defined is hard.

Then, for duals, we have



(2.7)

Then we have the identities:

(2.8)

Subsection 2.1.2

Definitions

Now let's define the things:

def 2.2 A category $\mathcal{C}$ is *additive* if we have that $\mathcal{C}(X, Y)$ is an abelian group where

- composition is biadditive, i.e.

$$(f + g) \circ h = f \circ h + g \circ h. \tag{2.9}$$

  note that the additions mean different things because they live in different hom-spaces.

- there is a *zero-object* O where in particular $C(O, O) \cong 0$.

- For all $X, Y \in \mathcal{C}$ there is some object $X \oplus Y$ and morphisms $i_X \in \mathcal{C}(X, X \oplus Y)$, $i_Y \in C(Y, X \oplus Y)$, $p_X \in \mathcal{C}(X \oplus Y, X)$, $p_Y \in \mathcal{C}(X \oplus Y, Y)$ subject to

$$p_X i_X = 1_X, \qquad p_Y i_Y = 1_Y \tag{2.10}$$
$$i_X p_X + i_Y p_Y = 1_{X \oplus Y} \tag{2.11}$$

  This object is unique up to unique isomorphism, so we might as well call this $X \oplus Y$. However note that given some X, Y, there might be many choices for the isomorphisms i and p.

sometimes this condition is split off where $p_X i_X = p_X \circ i_X$. when we mean tensor product, it will be written explicitly.

remark This category is connected, but between a lot of objects there is only the zero morphism.

lem 2.4 One can show that $p_Y i_X = 0_{X \to Y}$.

PROOF By inserting the identity:

$$p_Y i_X = p_Y(i_X p_X + i_Y p_Y) i_X \tag{2.12}$$
$$= p_Y i_X + p_Y i_X. \tag{2.13}$$

□

def 2.5 A morphism $e : X \to X$ is an *idempotent* if $e \circ e = e$.

Note that $(i_Y p_Y)(i_Y p_Y) = i_Y p_Y$ is idempotent.

def 2.6 An idempotent $e : X \to X$ *splits* if there exists some object A and maps $i : A \to X$ and $p : X \to A$ such that

$$pi = 1_A \qquad ip = e \tag{2.14}$$

def 2.7 *Idempotent complete* means all idempotents split.

so we are modelling something like vector spaces

def 2.8 $\mathcal{C}$ is linear if it's an additive category where each $\mathcal{C}(X, Y)$ is a (finite-dimensional complex) vector space and composition is bilinear.

biadditivity has been updated to bilinear

def 2.9 A linear monoidal category is linear, monoidal, and $\otimes : \mathcal{C}(X, Y) \times \mathcal{C}(Z, W)$ is linear in each component.

note that $\otimes$ is *not* necessarily a linear functor (it's bilinear).

Note: (adjective) functors on (adjective) categories preserve the structure of that category.

def 2.10 Let $\mathcal{C}$ be a linear category and $X \in \mathcal{C}$.

- $A \in \mathcal{C}$ is a subobject of X if there exists some $i : A \to X$ with a left inverse.

  Note that the zero object is a subobject of every object (inverse is zero-morphism). Objects are also subobjects of themselves (identity morphism). These are *trivial subobjects*.

- X is *simple* if it has no nontrivial subobjects.

- X is semisimple if it is isomorphic to a sum of simple objects.

  Note that to make this statement, we need associativity. In particular, $(X \oplus Y) \oplus Z \cong X \oplus (Y \oplus Z)$ (since these are unique up to unique isomorphism). Thus we can just pick one to work with and write $X \oplus Y \oplus Z$.

- X is *decomposable* if $X \cong Y \oplus Z$ where Y, Z are nontrivial (otherwise *indecomposable*).

  Note that simple and indecomposable are not equivalent (unless $\mathcal{C}$ is idempotent complete).

- $\mathcal{C}$ is *semisimple* if all objects are semisimple.

- $\mathcal{C}$ is finitely semisimple if it's semisimple and there is a finite number of isomorphism classes of simple objects.

  In physics, if a category represents particles w/ physical phenomena, then this is saying we just need to understand the objects.

this is not the most general definition, just an operational one which is good enough for our purposes.

these categories are very nice to work with! if we understand the finite number of simple objects, we understand everything.

Now we can finally understand def 2.1. It's still unclear why we want idempotent complete.

SECTION 2.2

Schur's lemma

We won't show this in the remaining time, but we'll give intuition.

The moral of the story: given a fusion category,

- X is simple iff $\mathrm{End}(X) \cong \mathbb{C}$
- for simple X, Y, we have Schur's lemma: $\mathcal{C}(X, Y) \cong \begin{cases} \mathbb{C} \\ 0 \end{cases}$

if they're isomorphic, there are maps (with structure of $\mathbb{C}$). otherwise there is just the zero map.

Let X be a simple object. We'll write $\oplus_X N_X X$ for the operation of taking direct sum of X a total of N_X times. Then:

$$\mathcal{C}(\oplus_X N_X X, \oplus_X M_X X) \cong \oplus_X M_{M_X, N_X}(\mathbb{C}) \quad (2.15)$$

so multimatrix object (direct sum of $M_X \times N_X$ matrices) are morphisms between semisimple objects.

Multifusion categories are all additive. We have kernels, cokernels, etc. and they all play well together. To show the second point, consider $f : X \to Y$ and we want to find all $g : X \to X$ such that $fg = 0$ (i.e. we want to find *right zeros* R^0). That is, we are finding a subspace of $\mathrm{End}(X)$.

(As one might hope) there exists an idempotent $e \in \mathcal{C}(X, X)$ which "projects" down to R^0. Note that if f is the zero map, $R^0 = \mathrm{End}(X)$. On the other hand, if $R^0 = 0$, then f is "injective" in some sense. Additionally, if Y is simple[2], f is "surjective".

[2] maybe X should also be?

If X is simple, $\mathrm{End}(X)$ is a division algebra.

Now we can ask: given a shape with pins on the boundary, what's the (vector) space of all diagrams we can draw (fixing the pins) in the bulk of the shape up to equivalence? In QM, we have a very geometric interpretation of writing down vector spaces using fusion categories. We'll get to this on Thursday.

By gluing circles inside holes, you get operads and Skein theory. We'll talk more about this and braiding tomorrow (Day 3).

q 2.11 **Luuk:** We have diagrams in beginning, but now we also have direct sums. Can we do this diagrammatically?
Kyle: no :'(. direct sums are denoted on the outside. taking direct sums of diagrams corresponds to taking direct sums of vector spaces corresponding to vector spaces. there can also be (hidden?) direct sums inside the diagram.

Day 3

DAY 3

Kyle has prepared five short talks. He will keep his eye on the time so that he can make time for the next speaker which is him for each of the talks.

We'll learn about pivotality, unitarity, sphericality. We'll learn about quantum mechanics. We'll talk about Skein theory. We'll talk about braiding (and Drinfel'd center).

SECTION 3.1

Fusion category

Last time we missed something in the definition of a fusion category.

def 3.1 A multi-fusion category $\mathcal{C}$ is a linear monoidal category which is

- finitely semisimple
- rigid

For us, semisimplicity will subsume idempotent complete. An *idempotent complete* category $\mathcal{C}$ is semisimple if for all $X \in \mathcal{C}$, $\mathrm{End}(X)$ (an algebra) is semisimple.

q 3.2 Does idempotent complete assume additive?
Kyle: We're assuming linearity which for us includes additivity.

Note that an algebra is semisimple if it's a multimatrix algebra. Thus

$$\mathrm{End}(X) \cong \oplus_k M_{N_k}(\mathbb{C}) \tag{3.1}$$

for positive integers $N_k > 0$.

thm 3.3 Let $\mathcal{C}$ be semisimple. Then all $X \in C$ have that $X \cong \oplus c_i$ where c_i are simple, and for any pair, either $c_i \cong c_j$ or they're distinct (i.e. $\mathcal{C}(c_i, c_j) = \mathcal{C}(c_j, c_i) = 0$, the vector space with one element).

Linear categories are connected because there are zero morphisms between any two objects. This also implies that simple object isomorphism classes are exactly the components of the category connected by nonzero morphisms.

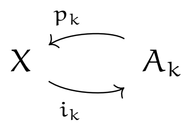
prop 3.4 $\mathcal{C}$ is semisimple. $X \in \mathcal{C}$ has $\mathrm{End}(X) \cong \mathbb{C}$ if and only if X is simple.

We'll build some tools so that we can get this result very quickly.

Now let's prove theorem 3.3. We'll be fast and loose with the isomorphism $X \cong \oplus_i c_i$.

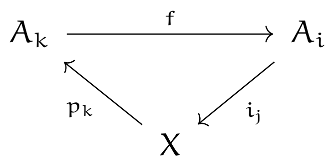
PROOF Let $\mathrm{End}(X) = \oplus_k M_{N_k}(\mathbb{C})$. This looks like a block diagonal matrix where blocks have dimension $N_k \times N_k$. Each block contains q_i which corresponds (under the isomorphism) to the identity in $M_{N_i}(\mathbb{C})$, i.e. we have a partition of unity.
Since they're mapped from the identity, each $q_k \circ q_k = q_k$ (i.e. it's

an idempotent). Since idempotents split, we can write $q_k = i_k \circ p_k$



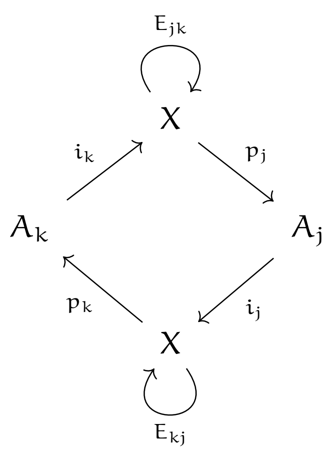
(3.2)

so that $p_k \circ i_k = i_{A_k}$ and $\oplus A_k \cong X$.
Suppose we have some morphism $A_k \xrightarrow{f} A_j$. We want to show that A_k and A_j are distinct. Taking E_{ij} to be the *matrix units* of the algebra, i.e. satsifying $E_{ij} q_{=} q_i E_{ij} = E_{ij}$. Note that f commutes with:



(3.3)

Since i_k has a left inverse and f has a right inverse, $i_j f_{p_k} = 0$ implies $f = 0$.
Now take $\mathrm{End}(X)$ to be simple, and take q_i to be the matrix which is only 1 in the ith slot on the diagonal. Now q_i splits, and the subobjects we get are $X \cong \oplus_k A_k$. Then we want to construct an isomorphism between A_ks, so by passing through X we have:



(3.4)

which explicitly has an inverse. Thus, they're all isomorphic. Now we must show they're simple. □

SECTION 3.2

Sphericality, unitality, pivotality

We can ask, can we identity $X^{\vee\vee} \cong X$ naturally? i.e. can we define a *pivotal structure* $p : 1_{\mathcal{C}} \xrightarrow{\sim} ((-)^{\vee})^{\vee}$?

It turns out that the answer is sometimes yes and sometimes no. It's an open question whether we always have one for fusion categories.

def 3.5 Let $\mathcal{C}$ be a linear category. A family of antilinear maps on morphisms $\dagger_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{C}(Y, X)$ is a *dagger structure* on $\mathcal{C}$ if e

$$((f)^\dagger)^\dagger = f \qquad (f \circ g)^\dagger = g^\dagger \circ f^\dagger \tag{3.5}$$

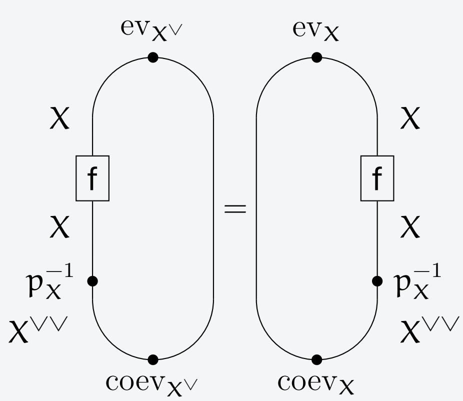
q 3.6 Can we define it as a contravariant functor from category to itself but acts as identity on objects?
Kyle: Yes. But it's not linear on hom-spaces.

def 3.7 $\dagger$ is *unitary* if there exists a norm on $\mathcal{C}(X, Y)$ such that $\dagger$ makes $\mathrm{End}(X)$ into a C^*-algebra.

That is, $\dagger$ looks like adjoints on morphisms. A *unitary fusion category* is a fusion category with a unitary (monoidal) dagger structure.

We'll talk about sphericality, and explain why unitary fusion categories automatically have a spherical pivotal structure.

def 3.8 A pivotal structure p is spherical if the *right trace* equals the *left trace*:



(3.6)

Unitarity is the first hint we're doing physics. We have not just a vector space, but a Hilbert space.

thm 3.9 A unitary fusion category (UFC) has spherical pivotal structure.

PROOF (sketch) The fundamental issue with left and right things being different is that coevs go in one direction for $X^\vee$ and the other way for X (we have curvy strings we can pull tight, one corresponds to X and the other corresponds to $X^\vee$).

Note that since dagger flips things, we can guess that

$$(\mathrm{coev}_X)^\dagger \overset{?}{=} \mathrm{ev}_{X^\vee} \tag{3.7}$$

Then, we can define these circle diagrams to be equal to the dimension of X. □

q 3.10 **Theo:** We only know $X^\vee$ up to unique isomorphism but not unique unitary isomorphism. If we pick a different object $X^\vee$ and write down a similar diagram, I could worry that the nonunitarity of the isomorphism, when I dagger things we don't get the same dagger.
Kyle: Yes at this point things are not yet spherical. But once we normalize things, it fixes things such that everything is spherical, possibly also unitary.

This is called "spherical" because if we draw these diagrams on the sphere, the left and right traces are the same (by moving the line along the back of the sphere).

SECTION 3.3

Quantum mechanics

Suppose we have a self-adjoint Hamiltonian operator H in a hilbert space $\mathcal{H}$. Operators $A \subseteq B(\mathcal{H})$ are bounded linear operators on $\mathcal{H}$. Observables consist of self-adjoint operators $x \in A$. We "measure" $X \to \lambda \in \mathrm{Spec}(X)$ by taking:

$$|\psi\rangle \overset{\text{measure}}{\longmapsto} |V_\lambda\rangle \tag{3.8}$$

where $X|V_\lambda\rangle = \lambda|V_\lambda\rangle$ with probability $p_\lambda = \frac{|\langle V_\lambda|\psi\rangle|^2}{\langle\psi|\psi\rangle}$. The Hamiltonian says

$$H|\psi_E\rangle = E|\psi_E\rangle, \tag{3.9}$$

where E is the energy of the state. Importantly, if we don't touch the system, measuring it again at a later time gives the same eigenvalue. There's some constant in the universe called $\hbar$ which we can formally expand in, and obtain time evolution:

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle. \tag{3.10}$$

Suppose we have an operator S (symmetry) which commutes with the Hamiltonian, and our state is an eigenstate, i.e. $S|\psi\rangle = \lambda|\psi\rangle$. After evolving in time, ψ will have the same eigenvalue of S (since it commutes with H). Oftentimes S is called "charge" or "flux".

Anything which is fixed under some symmetry action is kind of like a representation. Fusion categories are good at describing representations of things. In order to understand structures which are invariant in time, fusion categories are the tools we want to use.

ex 3.11 Suppose we have a Hilbert space $H = \otimes_{k=1}^{N} \mathbb{C}^2$. Since we're taking $\mathbb{C}^2$, we'll call each of these a qubit. Recall we have Pauli matrices. We'll call $X_i = I \otimes \cdots \otimes \underbrace{X}_{i\text{th}} \otimes \cdots \otimes I$ which is the operator localized on qubit i.
Let $H = -\sum_{i=1}^{N} Z_i Z_{i+1}$ (where the indices are mod N, i.e. periodic boundary conditions). Let $|\uparrow\rangle$ be the eigenvector of Z with $+1$ eigenvalue (and similarly $|\downarrow\rangle$ with -1 eigenvalue). The ground states (lowest energy state) are:

$$|g.s., \uparrow\rangle = \bigotimes_i |\uparrow\rangle \qquad |g.s., \downarrow\rangle = \bigotimes_i |\downarrow\rangle \tag{3.11}$$

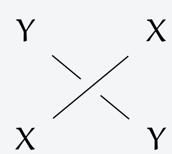
Whenever arrows change signs, we have a domain wall. There is no (local) operator which annihilates domain walls (hint that there's some kind of symmetry in the background preserving domain walls).

SECTION 3.4

Braiding and Skein theory

Tomorrow we'll see the Levin–Wen model. Give it a UFC and it gives you anyons.

def 3.12 A *braided monoidal category* has natural isomorphism $C_{X,Y} : X \otimes Y \to Y \otimes X$ which can be drawn:



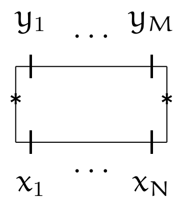
(3.12)

They satisfy the "hexagon equation" (basically Reidemeister moves) and the "triangle equation".

def 3.13 Given a monoidal category $\mathcal{C}$, the *Drinfel'd center* $Z(\mathcal{C})$ are objects (X, ψ) with $\psi \colon X \otimes (-) \to (-) \otimes X$ a natural isomorphism which respects the tensor product. Morphisms slide through the braiding.

Given a surface with marked boundary, e.g. D^2, the Skein module tells us all the ways to draw morphisms between external objects

$$S_{\mathcal{C}}(N, M, D^2) \simeq \bigoplus_{\text{ext. objs.}} \tag{3.13}$$



Note that in this case, the only diagram we can draw (up to isotopy) is the one which connects all external objects to a point in the bulk of the disk. However, in the case that the disk has a hole, then we can have multiple nonequivalent paths.

The most generic morphism in the Skein module of a tube (starting on inner circle and ending on outer circle) is the same as the hom space $\mathcal{C}(XY \to YX)$ where X is a guy in the middle of the cylinder. Then we can stack tubes on top of one another which gives us a multiplication which gets back to the same vector space, i.e. we have the *tube algebra*.

q 3.14 **Sameer:** If C is unitary fusion, is the braiding of $Z(C)$ unitary?
Kyle: Yes.

q 3.15 **Sophia:** We can't wiggle the guys on top to the guys on bottom of eq. (3.13)?
Kyle: No. We should put tags on the edge of the disk. If the category is pivotal we can put evaluation and coevaluation maps on the end to relate modules to each other.

DAY 4

Day 4

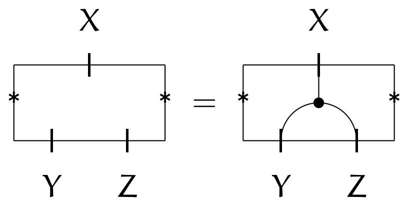
Physics time! We'll be doing lattice model TQFT!

SECTION 4.1

Skein theory

Let $\mathcal{C}$ be a unitary fusion category. $\mathrm{Irr}(\mathcal{C})$, is a finite set containing representatives of iso classes of simple objects. Suppose we have

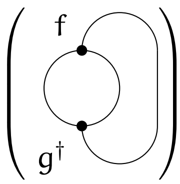
$X, Y, Z \in \mathrm{Irr}(\mathcal{C})$. We'll represent $\mathcal{C}(XY \to \mathbb{Z})$ by the picture



$$= S_{\mathcal{C}}(2, 1, D^2) \tag{4.1}$$

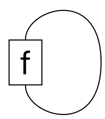
and

$$\langle f|g\rangle := \mathrm{Tr}\left(\ \right)\frac{1}{\sqrt{d_x d_y d_z}} \tag{4.2}$$



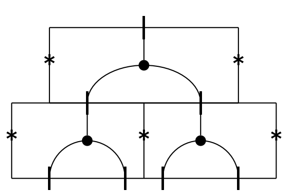
where recall that the trace is

$$\mathrm{tr}(f) = \tag{4.3}$$



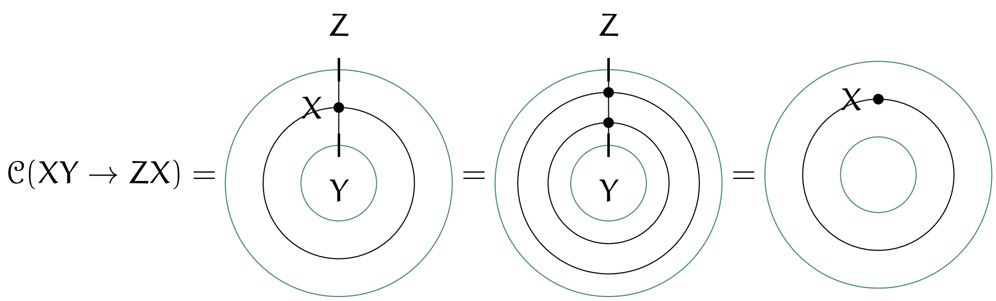
which makes sense using the spherical structure. Note that $\mathrm{End}(I) \cong \mathbb{C}$. Daggering a diagram flips it upside down and corresponds to complex conjugation on inner products. Thus $S(2, 1, D)^\dagger = S(1, 2, D)$.

We can compose by gluing diagrams with where outputs and inputs connect, and tensor diagrams together by gluing the tagged edge of the rectangle. Additionally, we can multiply $S(2, 1, D)$ with:



(4.4)

Note also that $S(M) \otimes S(N) \cong S(M \sqcup N)$. Recall the tube algebra

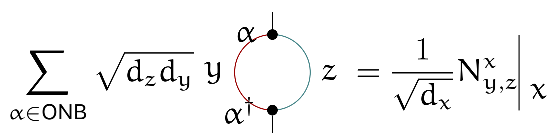


(4.5)

where in the last case, we took Y and Z to be the identity.

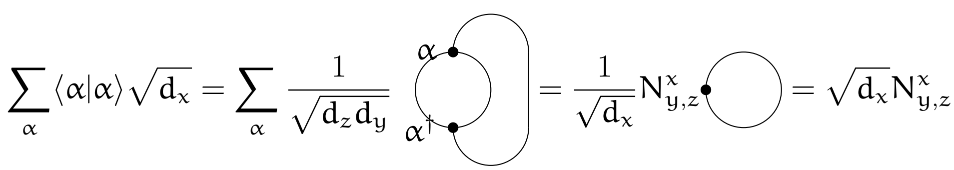
Note that we can also get an annulus with disks:

Additionally, tubes can be stacked with δ where input/outputs agree. Now how do we compose two rings into one? We can use the *bigon relation*:



(4.6)

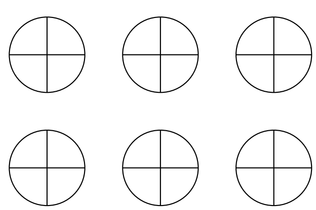
where $N_{y,z}^{x} := \dim(\mathcal{C}(yz \to x))$. Using this, we also have



(4.7)

When we're doing topological physics, we only care about universal properties. If we write down a toy model of a topological phase of matter, which is in the same equivalence class of a real physical thing (which is harder to solve), all of the universal (categorical) information will be the same between the two.

In lattice models, we have a Hilbert space which is a tensor product over all sites $\mathcal{H} = \otimes_v \mathcal{H}_v$. On each $\mathcal{H}_v$, we have an $S_{\mathcal{C}}(2, 2, D)$.



(4.8)

where there's a morphism at the center of each disk. Now, let $H = -\sum A_e - \sum B_p$ where $\mathrm{Spec}(A), \mathrm{Spec}(B) \leqslant 1$.

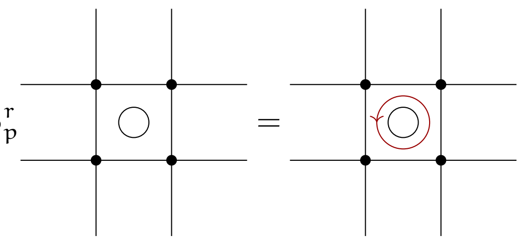
q 4.1 Each unitary fusion category gives you a Levin–Wen model. Two (Morita) equivalent UFCs give you different models but the same topological order. The order only depends on the center $Z(\mathcal{C})$.

An edge between two circles projects the two neighboring objects to be the same object. When objects are connected, we can also merge the circles, so after merging all circles, we effectively have discs on squares and obtain a grid with edges and plaquettes! Note that, $\prod_e A_e \neq 0$.

Now we'll take

$$B_p = \prod_e A_e(\tilde{B}_p) \prod_e A_e. \tag{4.9}$$

A state is a grid with edge labels and morphisms on corners. The way that $\tilde{B}_p$ acts on a grid with a hole is the same a wrapping an r-string around the hole:

$\tilde{B}_p^r$  (4.10)

The diagrams started going crazy so I stopped being able to TeX fast enough :'(. However, you can read more in [GHK+24].

By fixing boundary conditions, we've automatically figured out what's on the inside (holographic duality). By setting $\mathcal{C} = \mathrm{Vec}_{\mathbb{Z}_2}$, we get the toric code.

Day 5: Scenes from topological quantum theory

DAY 5

Today is bringing things together (less learning specifics). How are things related to topological physics (what is topological physics?), quantum information (which gives you funding).

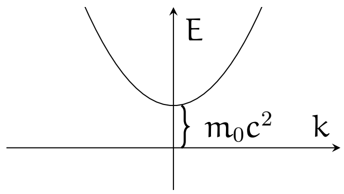
Here are three related fields: (topological) condensed matter[3],

a snap is a sound (a wave) but it's a very localized wavepacket. it's like a particle. the colder a system is, the more quantum things get. When things get cold, sound is quantized and becomes *phonons*. excitations in materials are *quasiparticles* (act like particles) but are

[3] study of systems with many particles, e.g. materials

sort of emergent from other things. but they're still fundamental: instead of using vacuum of universe, we use background of other things (Fermi sea).

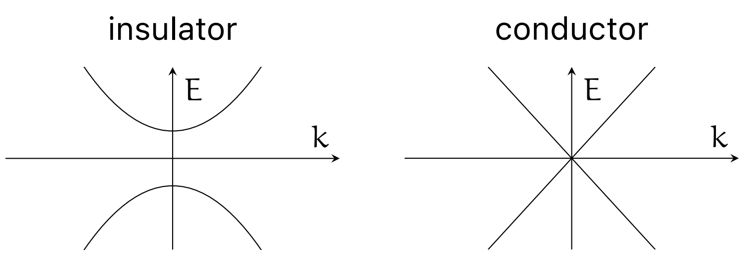
Recall $E = \gamma m_0 c^2$. in particular, $E(k)$ has a gap for a massive particle: it has a minimal energy m_0c^2.



(5.1)

same is true of anyons (form braided fusion categories). They form gapped systems, and you can't really get rid of an electron locally (you can't continuously shrink the energy of the system to zero). Similarly for us here, regardless of what we did in the school, (travelling, etc.) took a minimum amount of effort. after we get here, how much we do here is sort of a smooth spectrum, but staying home takes zero effort (and we have a gap).

Another example: consider a (homogenous) 2d lattice with translation symmetry. Since $[H, T_{x,y}] = 0$ we can simultaneously diagonalize both operators. And we can label various sectors of H by eigenvalues of T. If we're on a torus, then eigenvalues have some discrete labels (k_x, k_y) called *wavenumbers* (tell us about momentum of particles in system). An electron can hop between lattice sites through the Hamiltonian. In this system, the energy looks like:

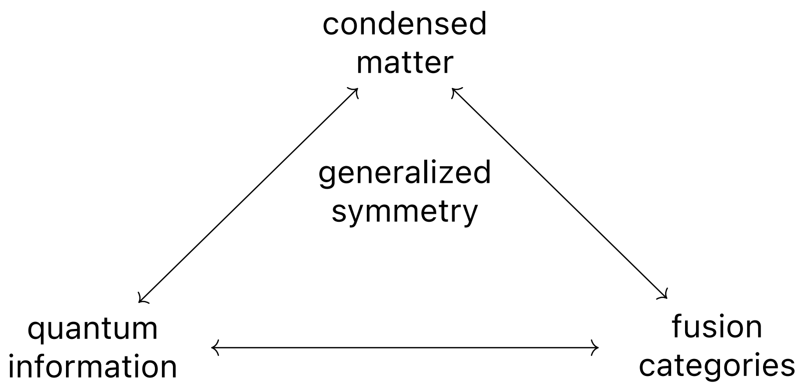


(5.2)

Then if we put electrons in the system, they first fill the bottom band. It takes a lot of energy to get the electrons to hop to the top band if there is a gap. This is called an insulator. In metals, on the other hand, there's no gap.

Some people put gallium arsenide in magentic field, which. Electrons make little circles, but on the boundary of the system, they can't make circles. So on the boundary there is a current and the system is effectively gapped. People want to understand which other systems have this topological behavior.

Now, we've been introduced to fusion category theory. We know there's a nice relationship between fusion categories and condensed matter. Fusion categories can tell us about quantum information. Quantum information and condensed matter are related e.g. through quantum error correcting codes. In the middle of this triangle (due to Sal) is generalized symmetry:



(5.3)

Let $\mathcal{H} = \otimes_{k=1}^{n} \mathbb{C}_k^2$ be our Hilbert space (n qubits) and S set of simple tensors of Pauli operators in $\mathcal{H}$ such that for $A, B \in S$ then $[A, B] = 0$. An $[N, k]$ *stabilizer code* is the set S where the dimension of the space where all $A \in S$ have $A = 1$ is 2^k. This space is called the *code space* L. For some $|\psi\rangle \in L$, we have $A|\psi\rangle = \pm|\psi'\rangle$ and this is called a *syndrome measurement*. If we output $+1$ for all of these measurements (if our code is good at correcting the errors we expect), then $A|\psi'\rangle = |\psi'\rangle$ and $|\psi'\rangle$ remains the same (no one messed with our system). If we get $+1$ on all of them, this is a *projection*.

Suppose someone makes an error. Suppose we have three qubits and $S = \{Z_1, Z_2, Z_2 Z_3\}$. The codespace $L = \mathrm{Span}(|\uparrow\uparrow\uparrow\rangle, |\downarrow\downarrow\downarrow\rangle)$ (these are all stabilized by S). If we only have bit-flip errors, we can only use two operators to figure out which error occurred. If your symmetries have extra structure (fusion categories), we can get Levin–Wen models.

Kitaev was not the first one to write it, but let's consider (what people call) Kitaev's toric code. Consider a square lattice with $\mathcal{H} = \otimes_e \mathbb{C}_e^2$ (a qubit at each edge on the lattice). Then consider the Hamiltonian

$$H = -\sum_v A_v - \sum B_p \tag{5.4}$$

(similar to last time, but $e \to v$). Let $A_v = Z_1 Z_2 Z_3 Z_4$ (choose a vertex, take the product of Z operators on the edges adjacent to the vertex) which takes $\pm\mathbb{1}$ (depending on how many up spins are surrounding the vertex). Then $B_p = X_{p_1} X_{p_2} X_{p_3} X_{p_4}$ (where X operators act on the four edges surrounding a plaquette). Note that $[A_v, A_{v'}] = [B_p, B_{p'}] =$

0. Then, note that:

$$[A_v, B_{p'}] = 0 \tag{5.5}$$

since either they share zero edges or two edges (X, Z anticommute, and each time we commute one past another on the same edge, we get a change in sign). A state is given by a lattice where we draw a line every time there's a down spin. If a vertex has an odd number of down spins next to it, $A_v = -\mathbb{1}$ (i.e. $A_v|\psi\rangle = -|\psi\rangle$). But note that if we require an even number of down spins at each vertex (i.e. if we take the subspace where $A_v = \mathbb{1}$) (*flux-free subspace*), then we get a *string net*. The ground state is the sum of these states(?).

If we take a torus, suppose we have a bunch of loops (where loops are made of down spins). The B_p operators take contractible loops to contractible loops. Note that acting with B_p operators always gives us a state with even $\mathbb{Z}_2$ monodromy. Similarly, there's the state with one vertical loop, one horizontal loop, and both a horizontal and vertical loop. Thus the codespace is four dimensional. One can show that trace of the product of the following projectors

$$\operatorname{tr}\left(\prod_v \left(\frac{1 + A_v}{2}\right) \prod_p \left(\frac{1 + B_p}{2}\right)\right) = \text{ground state degeneracy} \tag{5.6}$$

In general, this gives us 4^{genus}. These can be made into categories where, e.g. associators, are weighted sum of string net diagrams(?). So string nets are a beefed up version of toric code which can be used in QI as stabilizer codes.

Another way we can use string nets is by considering excited states. Let $A_v = ZZZZ$, $B_p = XXXX$. Then some operator $S^X = \prod_{i\in\gamma} X_i$ where γ is some path on the lattice. This commutes with all of the B_p operators, and all of the A_vs which are not at endpoints. In particular, no operator (A_v or B_p) can detect which path you took, only where the endpoints are (there are e excitations at endpoints). Similarly, on the dual lattice we can take a dual path $S^Z = \prod_{i\in\bar{\gamma}} Z_i$. In this case, we obtain m excitations. Thus we have $1, e, m, em$ (em is fused). Note that $S^X S^Z = -S^Z S^X$ and a braid gives $-\mathbb{1}$. So given a model, we can find a category (and vice versa).

q 5.1 **Luuk:** Is this related to definition of stabilizer code?
Kyle: Yes! $S = \{A_v, B_p\}$ and $L =$ ground state subspace. If you apply a string operator, can you find your original ground state again? if the string is small, then you can reconnect the ends of the string and everything is okay. But if the string is too long, then we don't know which way to reconnect (we've been tricked). If we know that the errors all happen in one place, then we can quarantine it (local

errors can be corrected).
Theo: Does this mean bigger errors are longer Hamming words?
Kyle: Here the Hamming distance between a codeword and an error word is not defined (it's only defined between codewords). Another way you can do this is say: imagine you can act on the whole system at once using local operators but you only have a small amount of time. Using perturbation theory you can still see that you can correct errors (Kitaev wrote about this).

Let $\omega : A \to \mathbb{C}$. We given a state $|\psi\rangle$ we can get a Hilbert space by defining an anyon with $\langle\psi|x|\psi\rangle = \omega(x)$. Suppose our states are objects between categories, and morphisms are things which look like

$$\omega(U \bullet U^\dagger) = \omega'(\bullet) \tag{5.7}$$

By thinking of all of the objects altogether, we get that every operator represents some morphism from this object to itself. If this is the identity object, we call this the universe.

$$\bullet \tag{5.8}$$

```
Kyle draws an object as a blue dot on the board, launches into
a prepared ``Pale Blue Dot'' monologue.

                              Kyle
            (ending the monologue)
        Everyone in our field has worked there, on
        this object suspended in a category.

                              Luuk
        We have one more lecture.
```

References

[GHK+24] David Green, Peter Huston, Kyle Kawagoe, David Penneys, Anup Poudel, and Sean Sanford. Enriched string-net models and their excitations. *Quantum*, 8:1301, March 2024.