INTRODUCTION TO FUSION CATEGORIES

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Abstract. In these notes, we introduce fusion categories and explain their relationship to topological physics.

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1. Rings and fusion rings

Definition 1.1. Let $(F, +)$ be a finitely generated abelian monoid. The triple $(F, +, \times)$ for $\times : F \times F \to F$ is called[1] a *fusion ring* if for all $a, b, c \in F$,

(1) the only element of F with an additive inverse is the additive identity,
(2)
$$a \times (b \times c) = (a \times b) \times c,$$
(3)
$$a \times (b + c) = a \times b + a \times c$$
and
$$(a + b) \times c = a \times c + b \times c.$$

A unital fusion ring has a multiplicative identity.

Although a fusion ring superficially looks similar to a ring, it is actually very different. Clearly, the definition is different, but the situation is slightly worse than that. Rings and fusion rings fundamentally describe different types of things. For example, fusion rings are good at describing the structure of ring bimodules in terms of their relative tensor product and direct sums.

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[1]There are other definitions one could use, but we choose this one.

In the fusion ring of $R-R$ bimodules, the equality of elements of the fusion ring means that the two elements are isomorphic as $R-R$ bimodules. Defining these isomorphisms in a consistent way is generally hard if you don't have full knowledge of the original ring R. If you only know the fusion ring, the best you can do is get a family of "coherence conditions." The solutions to these coherence conditions are examples of mathematical objects called monoidal categories. If one starts using fields and certain types of algebras as ingredients in a similar story, fusion categories will start showing up.

2. Monoidal categories

Definition 2.1. A *monoidal category* is a 6-tuple $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ where

(1) $\mathcal{C}$ is a category,
(2) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor,
(3) I is an object in $\mathcal{C}$ called the *unit object*,
(4) $\alpha : - \otimes (- \otimes -) \implies (- \otimes -) \otimes - : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a natural isomorphism called the *associator*,
(5) $\lambda : I \otimes - \implies - : \mathcal{C} \to \mathcal{C}$ is a natural isomorphism called the *left unitor*,
(6) $\lambda : - \otimes I \implies - : \mathcal{C} \to \mathcal{C}$ is a natural isomorphism called the *right unitor*.

These structures are subject to the following two coherence conditions given by commutative diagrams. In the following diagrams, we write $X \otimes Y$ as XY.

$$\begin{array}{ccccc} ((WX)Y)Z & \xrightarrow{\alpha_{W,X,Y}\otimes 1_Z} & (W(XY))Z & \xrightarrow{\alpha_{W,XY,Z}} & W((XY)Z) \\ \downarrow{\scriptstyle \alpha_{WX,Y,Z}} & & & & \downarrow{\scriptstyle 1_W \otimes \alpha_{X,Y,Z}} \\ (WX)(YZ) & & \xrightarrow{\qquad\alpha_{W,X,YZ}\qquad} & & W(X(YZ)) \end{array}$$

$$\begin{array}{ccc} (XI)Y & \xrightarrow{\alpha_{X,I,Y}} & X(IY) \\ & {\scriptstyle \rho_X \otimes 1_Y} \searrow \quad \swarrow {\scriptstyle 1_X \otimes \lambda_Y} & \\ & XY & \end{array}$$

Example 2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The endofunctor category $End(\mathcal{C})$ has a natural monoidal structure with the product given by functor composition.

In these notes, all vector spaces are complex and finite dimensional. For us, Vec is the category of such vector spaces. This convention includes representations and algebras as well.

Examples 2.3. Here are a few examples of monoidal categories that show up in our context frequently:

- Vec,
- the category of representations $\mathsf{Rep}G$ for a finite group G,
- the category of right (or left) modules Mod_A for a bialgebra A,
- the category of $A-A$ bimodules ${}_A\mathrm{Mod}_A$ for an algebra A with the relative tensor product $\boxtimes_A$.

For the remainder of this section, we refer to the monoidal categories $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$, $(\mathcal{D}, \bullet, J, \omega, l, r)$ simply as $\mathcal{C}$ and $\mathcal{D}$, respectively.

Definition 2.4. A *monoidal functor* from a monoidal category $\mathcal{C}$ to a monoidal category $\mathcal{D}$ is a triple (F, ϕ, ι) where

- $F : \mathcal{C} \to \mathcal{D}$ is a functor,
- $\phi : (F-) \bullet (F-) \implies F(- \otimes -)$ is a natural transformation,
- $\iota : J \to FI$ is a morphism.

The ι and the components of ϕ are called *structure morphisms*. These structures are subject to the following coherence conditions requiring that following diagrams commute:

$$\begin{array}{ccc}
(FX \bullet FY) \bullet FZ & \xrightarrow{\omega_{FX,FY,FZ}} & FX \bullet (FY \bullet FZ) \\
\downarrow{\scriptstyle \phi_{X,Y} \bullet 1_{FZ}} & & \downarrow{\scriptstyle 1_{FX} \bullet \phi_{Y,Z}} \\
F(X \otimes Y) \bullet FZ & & FX \bullet F(Y \otimes Z) \\
\downarrow{\scriptstyle \phi_{X \otimes Y, Z}} & & \downarrow{\scriptstyle \phi_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha_{X,Y,Z})} & F(X \otimes (Y \otimes Z))
\end{array}$$

$$\begin{array}{ccc}
FX \bullet J & \xrightarrow{1_{FX} \bullet \iota} & FX \bullet FI \\
\downarrow{\scriptstyle r_{FX}} & & \downarrow{\scriptstyle \phi_{X,I}} \\
FX & \xleftarrow[F(\rho_X)]{} & F(X \otimes I)
\end{array}$$

$$\begin{array}{ccc}
J \bullet FX & \xrightarrow{\iota \bullet 1_{FX}} & FI \bullet FX \\
\downarrow{\scriptstyle l_{FX}} & & \downarrow{\scriptstyle \phi_{I,X}} \\
FX & \xleftarrow[F(\lambda_X)]{} & F(I \otimes X)
\end{array}$$

A monoidal functor is called *strong* if the structure morphisms are invertible and *strict* if they are all identity morphisms.

Definition 2.5. A *monoidal equivalence* is a monoidal functor which is also an equivalence of categories.

Example 2.6. There is a monoidal functor from $\mathsf{Rep}G$ to Vec called the *forgetful functor*. This is the functor which assigns every representation to its underlying vector space. The structure morphisms are all the identity morphism.

Definition 2.7. Let $(F, \phi, \iota), (G, \theta, \eta) : \mathcal{C} \to \mathcal{D}$ be monoidal functors. A *monoidal natural transformation* $\gamma : F \implies G$ is a natural transformation such that the following diagrams commute:

$$\begin{array}{ccc}
FX \bullet FY & \xrightarrow{\gamma_X \bullet \gamma_Y} & GX \bullet GY \\
\downarrow{\scriptstyle \phi_{X,Y}} & & \downarrow{\scriptstyle \theta X,Y} \\
F(X \otimes Y) & \xrightarrow{\gamma_{X \otimes Y}} & G(X \otimes Y)
\end{array}$$

$$\begin{array}{ccccc}
 & & J & & \\
 & \swarrow{\scriptstyle \eta} & & \searrow{\scriptstyle \iota} & \\
FI & & \xrightarrow{\gamma_I} & & GI
\end{array}$$

For the remainder of this document, we will exclude associator and unitor labels in diagrams where these labels are clear from context.

Definition 3.1. Let $\mathcal{C}$ be a monoidal category and let $X \in \mathcal{C}$. An object X^* is called *a left dual of* X if there exist $\mathrm{ev}_X \in \mathcal{C}(X^\vee \otimes X, I)$ and $\mathrm{coev}_X \in \mathcal{C}(I, X \otimes X^\vee)$ (notice the order of X and X^*) such that the following diagrams commute:

$$\begin{array}{ccc}
X & \xrightarrow{\quad 1_X \quad} & X \\
\downarrow & & \uparrow \\
I \otimes X & & X \otimes I \\
\downarrow{\scriptstyle \mathrm{coev}_X \otimes 1_X} & & {\scriptstyle 1_X \otimes \mathrm{ev}_X}\uparrow \\
(X \otimes X^\vee) \otimes X & \longrightarrow & X \otimes (X^\vee \otimes X)
\end{array}$$

$$\begin{array}{ccc}
X^\vee & \xrightarrow{\quad 1_X \quad} & X \\
\downarrow & & \uparrow \\
X^\vee \otimes I & & I \otimes X^\vee \\
\downarrow{\scriptstyle 1_{X^\vee} \otimes \mathrm{coev}_X} & & {\scriptstyle \mathrm{ev}_X \otimes 1_{X^\vee}}\uparrow \\
X^\vee \otimes (X \otimes X^\vee) & \longrightarrow & (X^\vee \otimes X) \otimes X^\vee
\end{array}$$

Likewise, ${}^\vee X$ is *a right dual of* X if there exist $\mathrm{ev}'_X \in \mathcal{C}(X \otimes {}^\vee X, I), \mathrm{coev}'_X \in \mathcal{C}(I, {}^\vee X \otimes X)$ such that the following diagrams commute:

$$\begin{array}{ccc}
X & \xrightarrow{\quad 1_X \quad} & X \\
\downarrow & & \uparrow \\
X \otimes I & & I \otimes X \\
\downarrow{\scriptstyle 1_X \otimes \mathrm{coev}'_X} & & {\scriptstyle \mathrm{ev}'_X \otimes 1_X}\uparrow \\
X \otimes ({}^\vee X \otimes X) & \longrightarrow & (X \otimes {}^\vee X) \otimes X
\end{array}$$

$$\begin{array}{ccc}
X^\vee & \xrightarrow{\quad 1_X \quad} & X \\
\downarrow & & \uparrow \\
X^\vee \otimes I & & I \otimes X^\vee \\
\downarrow{\scriptstyle 1_{X^\vee} \otimes \mathrm{coev}_X} & & {\scriptstyle \mathrm{ev}_X \otimes 1_{X^\vee}}\uparrow \\
X^\vee \otimes (X \otimes X^\vee) & \longrightarrow & (X^\vee \otimes X) \otimes X^\vee
\end{array}$$

Definition 3.2. A category is called rigid if all objects have a left and right dual.

Fact 3.3. Left and right duals are unique up to unique isomorphism. We are therefore justified in saying "the" left and right dual.

Example 3.4. In Hilb, notice that the left dual of a Hilbert space $\mathcal{H}$ is its dual space $\mathcal{H}^\vee$ which is the space of linear functionals $\mathcal{H} \to \mathbb{C}$. The evaluation morphism is literally the evaluation of the functional $f \in \mathcal{H}^\vee$ on a vector $v \in \mathcal{H}$, $f \otimes v \mapsto f(v)$. On the other hand, if $\{e_i\}$ is an orthonormal basis for $\mathcal{H}$, then we may take the coevaluation map to be

$$z \mapsto z \sum_i e_i \otimes e_i^\vee$$

where $e_i^\vee(e_j) = \delta_{i,j}$. The above coevaluation map is basis independent.

4. Linear categories

4.1. **Linear structure.**

Definition 4.1. A category $\mathcal{C}$ is called *additive* if each Hom set in $\mathcal{C}$ is given the structure of an abelian group and obeys the following properties:

(M1) Composition of morphisms is a biadditive,
(M2) there is a zero object $0 \in \mathcal{C}$ such that $\mathcal{C}(0,0) = 0$ (the trivial group),
(M3) for every two objects, $X, Y \in \mathcal{C}$, there exists an object $X \oplus Y$ and morphisms

$$\begin{aligned} i_X &\in \mathcal{C}(X, X \oplus Y) \\ i_Y &\in \mathcal{C}(Y, X \oplus Y) \\ p_X &\in \mathcal{C}(X \oplus Y, X) \\ p_Y &\in \mathcal{C}(X \oplus Y, Y) \end{aligned}$$

such that
(IP1) $p_X i_X = 1_X$
(IP2) $p_Y i_Y = 1_Y$
(IP3) $i_X p_X + i_Y p_Y = 1_{X \oplus Y}$.

Lemma 4.2. *Let $\mathcal{C}$ be additive and let $X, Y \in \mathcal{C}$. Then $i_X \in \mathcal{C}(X, X \oplus Y)$ and $p_Y \in \mathcal{C}(X \oplus Y, Y)$ have $p_Y i_X = 0$. Similarly, $p_X i_Y = 0$.*

Proof. Since $i_X p_X + i_Y p_Y = 1_{X \oplus Y}$,

$$p_Y i_X = p_Y (i_X p_X + i_Y p_Y) i_X = p_Y i_X + p_Y i_X.$$

Therefore, $p_Y i_X = 0$. Similarly, $p_X i_Y = 0$. □

Lemma 4.3. *Let $\mathcal{C}$ be an additive category and let $X, Y \in \mathcal{C}$ be non-zero objects. Then $i_X p_X \neq 0$ and $i_Y p_Y \neq 0$.*

Proof. In a linear category, a zero morphism composed with any other morphism is also a zero morphism. However,

$$p_X (i_X p_X) i_X = 1_X.$$

It is possible to show that an object $A \in \mathcal{C}$ is a zero object if and only if $End(A) = 0$. Therefore, the non-trivial object X has $1_X \neq 0$. Therefore $i_X p_X \neq 0$. The same argument holds for why $i_Y p_Y \neq 0$. □

Proposition 4.4. *Let $\mathcal{C}$ be additive and let $X, Y \in \mathcal{C}$. Then $X \oplus Y$ is unique up to isomorphism. This isomorphism is unique in the sense that there is a unique isomorphism which map the inclusions and projections of one direct sum to another.*

Proof. Suppose Z, W obey the axioms for the direct sum of X, Y via the morphisms $i_X^Z, i_Y^Z, p_X^Z, p_Y^Z$ and $i_X^W, i_Y^W, p_X^W, p_Y^W$. Then

$$i_X^W p_X^Z + i_Y^W p_Y^Z \in \mathcal{C}(Z, W) \qquad i_X^Z p_X^W + i_Y^Z p_Y^W \in \mathcal{C}(W, Z).$$

These morphisms obey

$$\begin{aligned}(i^W_X p^Z_X + i^W_Y p^Z_Y)(i^Z_X p^W_X + i^Z_Y p^W_Y) &= i^W_X p^Z_X i^Z_X p^W_X + i^W_Y p^Z_Y i^Z_X p^W_X + i^W_X p^Z_X i^Z_Y p^W_Y + i^W_Y p^Z_Y i^Z_Y p^W_Y \\ &= i^W_X p^W_X + 0 + 0 + i^W_Y p^W_Y \\ &= 1_W.\end{aligned}$$

Similarly,

$$(i^Z_X p^W_X + i^Z_Y p^W_Y)(i^W_X p^Z_X + i^W_Y p^Z_Y) = 1_Z.$$

Therefore $Z \cong W$.

We leave it to the reader to prove that if there is some isomorphism $\phi \in \mathcal{C}(Z, W)$ such that

$$\phi i^Z_X = i^W_X, \quad \phi i^Z_Y = i^W_Y, \quad p^Z_X \phi = p^W_X, \quad p^Z_Y \phi = p^W_Y,$$

then $\phi = i^W_X p^Z_X + i^W_Y p^Z_Y$. □

Remark 4.5. The above discussion of uniqueness is fairly subtle. Certainly, there are many isomorphisms in Vec from $\mathbb{C} \oplus \mathbb{C}$ to itself. However, once one makes a choice of inclusions and projections, only the identity morphism preserves these structures.

Remark 4.6. Proposition 4.4 justifies our notation which implicitly picks a distinguished object to call $X \oplus Y$ for each X and Y. For this reason, we also choose to pick these representative objects so that $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$. Also note that $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor.

Definition 4.7. Let k be a field. An additive category $\mathcal{C}$ is called k-linear if each Hom group is further endowed with the structure of a vector space over k and morphism composition is bilinear.

In these notes, all linear categories are taken to be $\mathbb{C}$-linear. Recall that we also take all vector space to be finite dimensional.

Definition 4.8. Let $\mathcal{C}, \mathcal{D}$ be linear categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *linear* if it is linear on Hom spaces.

Definition 4.9. Let $\mathcal{C}$ be a monoidal category with monoidal product $\otimes$ and also let $\mathcal{C}$ be linear. We say that $\mathcal{C}$ is a linear monoidal category if the maps $\otimes : \mathcal{C}(X, Y) \times \mathcal{C}(Z, W) \to \mathcal{C}(X \otimes Z, Y \otimes W)$ are linear in both arguments.

Definition 4.10. A linear monoidal category is one for which the monoidal product is a linear functor.

Fact 4.11. In an additive category $\mathcal{C}$ with objects $X, Y \in \mathcal{C}$ such that $\mathcal{C}(X, Y) \neq 0$, the 0 morphism in $\mathcal{C}(X, Y)$ is not invertible.

Definition 4.12. Let $\mathcal{C}$ be an additive category. A morphism $e \in \mathcal{C}(X, X)$ is called an *idempotent* if $e \circ e = e$. We say that the idempotent e *splits* if there exists some object A and morphisms $p \in \mathcal{C}(X, A), i \in \mathcal{C}(A, X)$ such that $e = i \circ p$ and $p \circ i = 1_A$. The category $\mathcal{C}$ is called *idempotent complete* if all idempotents in $\mathcal{C}$ split.

Exercise 4.13. Let $\mathcal{C}$ be idempotent complete. Prove that an object in $\mathcal{C}$ is simple if and only if it is indecomposable.

4.2. **Simplicity.** We have tailored the following definition for our purposes.

Definition 4.14. Let $\mathcal{C}$ be an additive category with objects $X, Y \in \mathcal{C}$. If there is a morphism $x \in \mathcal{C}(X, Y)$ with a left inverse, then we say that X is a *subobject* of Y. Note that 0 is a subobject of every object and every object is a subobject of itself. A subobject Y of X is called a *trivial* subobjects if $Y \cong 0$ or $Y \cong X$.

Definition 4.15. In the context of linear categories:

- An object is *simple* if it has no non-trivial subobjects.
- An object is *decomposable* if it is isomorphic to a direct sum of two non-zero objects.
- An object is *semisimple* if it is isomorphic to a direct sum of simple objects.
- An idempotent complete linear category $\mathcal{C}$ is called *semisimple* if for all $X \in \mathcal{C}$, $End(X)$ is a semisimple algebra.
- A semisimple category is *finitely semisimple* if it is semisimple and has a finite number of isomorphism classes of simple objects.

Lemma 4.16. *Let $\mathcal{C}$ be a linear category with a non-simple object $X \in \mathcal{C}$. Then we know there exists some non-trivial subobject A with morphisms $i \in \mathcal{C}(A, X)$ and $p \in \mathcal{C}(X, A)$ such that $i \circ p = 1_A$. We have that $p \circ i \neq 0$ and $p \circ i \neq 1_X$.*

Proof. The proof follows similarly to Lemma 4.3 to show that $p \circ i \neq 0$. We also know that $p \circ i \neq 1_X$ since X and A are not isomorphic. □

Definition 4.17. Let $\mathcal{C}$ be a linear category. Objects $X, Y \in \mathcal{C}$ are *distinct* if $\mathcal{C}(X, Y) \cong 0 \cong \mathcal{C}(Y, X)$.

Lemma 4.18. *Let $\mathcal{C}$ be idempotent complete and let $A \in \mathcal{C}$. If $e \in End(A)$ is a non-trivial idempotent ($e \neq 0, e \neq 1_X$), then A is decomposable.*

Proof. Let $e \in End(X)$ be a non-trivial idempotent. Then $1 - e$ is also a non-trivial idempotent. Since every idempotent splits in $\mathcal{C}$, there exists objects A, B along with

$$i_X \in \mathcal{C}(X, A)$$
$$i_Y \in \mathcal{C}(Y, A)$$
$$p_X \in \mathcal{C}(A, X)$$
$$p_Y \in \mathcal{C}(A, Y)$$

which satisfy the usual inclusion/projection relations making $X \oplus Y \cong A$. □

Lemma 4.19. *Let $\mathcal{C}$ be linear and let $X, Y \in \mathcal{C}$ be non-zero objects. Then $dim(End(X \oplus Y)) \geq 2$. In particular, $End(X \oplus Y) \ncong \mathbb{C}$.*

Proof. First, notice that since X and Y are not zero objects, Lemma 4.3 implies that $i_X \circ p_X \neq 0$ and $i_Y \circ p_Y \neq 0$. Lemma 4.2 tells us that $i_X \circ p_X \circ i_Y \circ p_Y = 0$. Since two non-zero elements of $\mathbb{C}$ cannot multiply to 0, we know that $End(X \oplus Y) \ncong \mathbb{C}$. Therefore, $dim(End(X \oplus Y)) \geq 2$. □

Proposition 4.20. *Let $\mathcal{C}$ be idempotent complete. Then an object $X \in \mathcal{C}$ is simple if and only if it is indecomposable.*

Proof. It is clear that simplicity implies indecomposability, so we only prove the other direction. Suppose $X \in \mathcal{C}$ is not simple. Then there exists some non-trivial subobject A with $i \in \mathcal{C}(A, X)$ and $p \in \mathcal{C}(X, A)$ such that $p \circ i = 1_A$. Since A is non-trivial, Lemma 4.16 tells us that $i \circ p \neq 0$ and $i \circ p \neq 1_X$. This implies that $i \circ p$ is a non-trivial idempotent. By Lemma 4.18, we have that X is decomposable. □

Theorem 4.21. *Let $\mathcal{C}$ be semisimple and let $X \in \mathcal{C}$. Then X is simple if and only if $End(X) \cong \mathbb{C}$.*

Proof. Suppose that $End(X) \ncong \mathbb{C}$. Since $End(X)$ is semisimple,

$$End(X) \cong \bigoplus_{i=1}^{\Gamma} M_{N_i}(\mathbb{C})$$

where N_i are positive integers. Since $End(X) \ncong \mathbb{C}$, we have that

$$N := \sum_{i=1}^{\Gamma} N_i \geq 2.$$

Now let $p \in End(X)$ be the idempotent corresponding to the matrix unit $E_{11} \in M_{N_1}(\mathbb{C})$ (the matrix with a 1 in the top left corner and is zero everywhere else). Since $N \geq 2$ and $p \neq 0$, we know that p is a non-trivial idempotent. By Lemma 4.18, we know that X must be decomposable. Proposition 4.20 then implies that X is not simple.

If X is not simple, proposition 4.20 implies that X is decomposable. Lemma 4.19 then implies that $End(X) \ncong \mathbb{C}$. □

Theorem 4.22. *Let $\mathcal{C}$ be a semisimple category. Then for every object $X \in \mathcal{C}$, there exist simples c_i such that*

$$X \cong \bigoplus_i c_i$$

such that for all c_i, c_j, $c_i \cong c_j$ or c_i and c_j are distinct. In particular, every object in a semisimple category is semisimple.

Proof. Let $X \in \mathcal{C}$. Then $End(X) \cong \bigoplus_i M_{N_i}(\mathbb{C})$. Therefore these exist $q_i \in End(X)$ which are projections to the ith simple component of $End(X)$ so that

$$1_X = \sum_i q_i.$$

Using the same reasoning as many other proofs in this subsection, each of these summands correspond to different subobjects of X whose direct sum is isomorphic to X. Let A, B be two such subobjects. Consider a morphism $f : A \to B$. Then we have that $f = p_B \circ i_B \circ f \circ p_A \circ i_A$. However, we know from the simple decomposition of $End(X)$ that $i_B \circ f \circ p_A = 0$, so $f = 0$.

We now just have to prove that the summands of X decompose into isomorphic simple objects. It is therefore sufficient to now just take X to be simple so that $End(X) \cong M_n(\mathbb{C})$. Take $q_k \in End(X)$ to now be the idempotent corresponding to the diagonal matrix with a single 1 in the kth place of the diagonal and 0's everywhere else. Each idempotent then corresponds to a summand A_k. Now take $E_{ij} \in End(X)$ to correspond to the standard ijth matrix unit so that $E_{ij} = q_i \circ E_{ij} \circ q_j$. Finally, we have from A_k to

A_j given by $p_{A_j} \circ E_{jk} \circ i_{A_k}$. Note that $q_k = i_{A_k} \circ p_{A_k}$ and $q_k = E_{kj} \circ E_{jk}$. The inverse of $p_{A_j} \circ E_{jk} \circ i_{A_k}$ is $p_{A_k} \circ E_{kj} \circ i_{A_j}$ which proves $A_j \cong A_k$.

□

4.3. **Unitarity.**

Definition 4.23. Let $\mathcal{C}$ be a linear category. A *dagger structure* $\dagger$ on $\mathcal{C}$ is a collection of anti-linear maps $\dagger_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{C}(Y,X)$ such that for all $X, Y, Z \in \mathcal{C}$,

(1) $\dagger_{Y,X} \circ \dagger_{X,Y} = id_{\mathcal{C}(X,Y)}$,
(2) for $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$, $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

Note that the above axioms introduce a convenient notation. A linear category equipped with a dagger structure is called a *dagger* category.

Fact 4.24. For a linear category $\mathcal{C}$ and object $X \in \mathcal{C}$, $End(X)$ is an algebra. A dagger structure on $\mathcal{C}$ then makes $End(X)$ into a $*$-algebra by setting $* = \dagger_{X,X}$.

Definition 4.25. Let $\mathcal{C}$ be a dagger category. If for each $X \in \mathcal{C}$ $End(X)$ is a unitary algebra, then we say that $\mathcal{C}$ is *unitary*.

5. Braided monoidal categories

Definition 5.1. Let $\mathcal{C}$ be a monoidal category. A *braiding* for an object $X \in \mathcal{C}$ is a natural isomorphism $\psi : - \otimes X \to X \otimes -$ such that the following diagram commutes:

$$\begin{array}{ccccc} & Y \otimes (Z \otimes X) & \xrightarrow{1_Y \otimes \psi_Z} & Y \otimes (X \otimes Z) & \\ \nearrow & & & & \searrow \\ (Y \otimes Z) \otimes X & & & & (Y \otimes X) \otimes Z \\ \searrow^{\psi_{Y \otimes Z}} & & & & \nearrow_{\psi_Y \otimes 1_Z} \\ & X \otimes (Y \otimes Z) & \longrightarrow & (X \otimes Y) \otimes Z & \end{array}$$

Definition 5.2. For a moniodal category $\mathcal{C}$, we call a family of moprhisms $c_{X,Y}$ for each $X, Y \in \mathcal{C}$ a braiding on $\mathcal{C}$ (and $\mathcal{C}$ a braided monoidal category) if $c_{-,X}$ and $c^{-1}_{X,-}$ are both braidings on $X \in \mathcal{C}$. **TODO: check and add in unitor stuff**

Definition 5.3. Suppose $\mathcal{C}$ is a monoidal category. The *Drinfeld center* $Z(\mathcal{C})$ of $\mathcal{C}$ is a category whose objects are given by all pairs (X, ψ) where $X \in \mathcal{C}$ and ψ is a braiding for X. The morphisms $f \in Z(\mathcal{C})((X,\psi),(Y,\phi))$ are morphisms $f \in \mathcal{C}(X,Y)$ such that the following diagram commutes

$$\begin{array}{ccc} ZX & \xrightarrow{1_Z \otimes f} & ZY \\ \downarrow{\scriptstyle \psi_Z} & & \downarrow{\scriptstyle \phi_Z} \\ XZ & \xrightarrow{f \otimes 1_Z} & YZ \end{array}$$

Definition 5.4. Let $\mathcal{C}$ be a monoidal category with product $\otimes$. Then we define the functor (using the same symbol unfortunately) $\otimes : Z(\mathcal{C}) \otimes Z(\mathcal{C}) \to Z(\mathcal{C})$ on objects $(X,\psi), (Y,\phi)$ as $(X,\psi) \otimes (Y,\phi) := (X \otimes Y, \alpha)$ where

$$\alpha_Z := (1_X \otimes \phi_Z) \circ (\psi_Z \otimes 1_Y).$$

We leave it to the reader to check that $(X,\psi)\otimes(Y,\phi)\in Z(\mathcal{C})$. For morphisms,

$$((X,\psi)\xrightarrow{f}(Y,\phi))\otimes((Z,\eta)\xrightarrow{g}(Y,\gamma)) := (X\xrightarrow{f}Y)\otimes(Z\xrightarrow{g}Y).$$

6. Fusion categories

There are many ways to define fusion categories. Some definitions show case many other more general types of categories. Here we have opted for a more direct approach.

Definition 6.1 ((Multi-)Fusion category)**.** A linear monoidal category $\mathcal{C}$ is called *multi-fusion* if it is

- finitely semisimple (recall that semisimple subsumes idempotent complete),
- rigid.

We say that $\mathcal{C}$ is *fusion* if $End(I)\cong\mathbb{C}$.

Remark 6.2. In the definition of a fusion category, $End(I)\cong\mathbb{C}$ subsumes the axiom that I is simple when assuming the other axioms. This is because $I = X\oplus Y$ means that $End(I)$ is at least 2 dimensional.

See the following for notes on the relationship between unitarity and sphericity. https://people.math.osu.edu/penneys.2/8800/Notes/FusionCats.pdf

6.1. Tube algebra.

Definition 6.3. Let $\mathcal{C}$ be a unitary fusion category. For $f,g\in\mathcal{C}(x,y\otimes z)$, define $\langle f|g\rangle = Tr(f^\dagger g)$ using the trace Tr.

Proposition 6.4. *The definition of* $\langle\cdot|\cdot\rangle$ *from definition 6.3 is an inner product.*

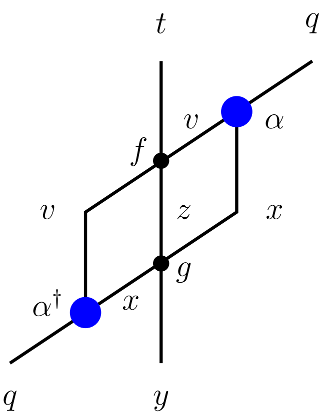
Proof. **TODO:** □

Definition 6.5. Let $\mathcal{C}$ be a unitary fusion category. As a vector space, we define $Tube(\mathcal{C})$ as

$$Tube(\mathcal{C}) := \bigoplus_{x,y,z\in Irr(\mathcal{C})} \mathcal{C}(x\otimes y, z\otimes x)$$

where the sum is taken over representatives of isomorphism classes of simple objects. Call for $x,y,z\in Irr(\mathcal{C})$, $Tube(\mathcal{C})_{x,y,z} := \mathcal{C}(x\otimes y, z\otimes x)$. We now define the multiplication $\cdot : Tube(\mathcal{C})\otimes Tube(\mathcal{C})\to Tube(\mathcal{C})$. For $w\neq z$, we take $Tube(\mathcal{C})_{v,w,t}Tube(\mathcal{C})_{x,y,z}=0$. Let B_q be an orthonormal basis for $\mathcal{C}(v\otimes x, q)$ for $q\in Irr(\mathcal{C})$ using the inner product from Definition 6.3. Then for $f\in Tube(\mathcal{C})_{v,z,t}$ and $g\in Tube(\mathcal{C})_{x,y,z}$, we define

$$f\cdot g := \sum_{q\in Irr(\mathcal{C})}\sum_{\alpha\in B_q}\sqrt{\frac{d_q}{d_v d_x}}$$



References