## Introduction to algebraic K-theory Problem set 1

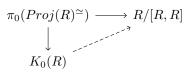
- **Ex. 1.** Recall that the group completion functor  $(-)^{\text{gp}}$ : CMon  $\rightarrow$  Ab is a left adjoint to the inclusion Ab  $\rightarrow$  CMon.
  - 1. Show that the unit map  $M \to M^{\text{gp}}$  is injective if an only if for all a, b, c in M, it holds that a + c = b + c implies a = b.
  - 2. Let X be a topological space. Show that  $Map(X, \mathbb{N})^{gp} \simeq Map(X, \mathbb{Z})$ , where Map(X, Y) denotes the continuous maps  $f: X \to Y$ , and  $\mathbb{N}$  and  $\mathbb{Z}$  are considered as discrete topological spaces.
- **Ex. 2.** Show that {iso. classes of countably generated projective *R*-modules}<sup>gp</sup> =  $\{0\}$ .

*Hint:* Use a Hilbert hotel like argument.

## Ex. 3. The Hattori-Stallings trace.

Let R be an arbitrary ring, and M a left R-module.

- 1. Show that the hom  $\operatorname{Hom}_R(M, R)$  has a right *R*-module structure.
- 2. Show that there is a map ev:  $\operatorname{Hom}_R(M, R) \otimes_R M \to R/[R, R]$  induced by the evaluation map ev:  $\operatorname{Hom}_R(M, R) \otimes_{\mathbb{Z}} M \to R$ .
- 3. Using the previous parts, construct the canonical trace as below



**Note:** The group R/[R, R] is also known as  $HH_0(R)$ , the zeroth Hochschild homology of R. The *Dennis trace*, which is a fundamental tool for computations in algebraic K-theory, is a generalization of the Hattori-Strallings trace to higher K-theory,  $K_*(R) \to HH_*(R)$ .