Introduction to algebraic K-theory
Problem set 1

Ex. 1. Recall that the group completion functor $(-)^{\mathrm{gp}}\colon \mathrm{CMon} \to \mathrm{Ab}$ is a left adjoint to the inclusion $\mathrm{Ab} \to \mathrm{CMon}$.

1. Show that the unit map $M \to M^{\mathrm{gp}}$ is injective if an only if for all a, b, c in M, it holds that $a + c = b + c$ implies $a = b$.
2. Let X be a topological space. Show that $Map(X, \mathbb{N})^{\mathrm{gp}} \simeq Map(X, \mathbb{Z})$, where $Map(X, Y)$ denotes the continuous maps $f\colon X \to Y$, and $\mathbb{N}$ and $\mathbb{Z}$ are considered as discrete topological spaces.

Ex. 2. Show that $\{\text{iso. classes of countably generated projective } R\text{-modules}\}^{\mathrm{gp}} = \{0\}$.

Hint: Use a Hilbert hotel like argument.

Ex. 3. The Hattori-Stallings trace.

Let R be an arbitrary ring, and M a left R-module.

1. Show that the hom $\mathrm{Hom}_R(M, R)$ has a right R-module structure.
2. Show that there is a map $\mathrm{ev}\colon \mathrm{Hom}_R(M, R) \otimes_R M \to R/[R, R]$ induced by the evaluation map $\mathrm{ev}\colon \mathrm{Hom}_R(M, R) \otimes_{\mathbb{Z}} M \to R$.
3. Using the previous parts, construct the canonical trace as below

$$\begin{array}{ccc} \pi_0(Proj(R)^{\simeq}) & \longrightarrow & R/[R,R] \\ \downarrow & \nearrow & \\ K_0(R) & & \end{array}$$

Note: The group $R/[R, R]$ is also known as $HH_0(R)$, the zeroth Hochschild homology of R. The *Dennis trace*, which is a fundamental tool for computations in algebraic K-theory, is a generalization of the Hattori-Strallings trace to higher K-theory, $K_*(R) \to HH_*(R)$.