Introduction to algebraic K-theory
Problem set 3

Ex. 1. Let $\mathcal{E}$ be an exact category. Without using the additivity theorem, show that the map
$$K_0(e(\mathcal{E})) \to K_0(\mathcal{E}) \times K_0(\mathcal{E})$$
induced by the functor $(s, q)\colon e(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ seen in the lecture, is always an isomorphism.

Ex. 2. Let $\mathcal{E}$ be an exact category. Show that the span $X \twoheadleftarrow Z \rightarrowtail Y$ is an isomorphism in $Q\mathcal{E}$ if and only if $p\colon Z \to X$ and $i\colon Z \rightarrowtail Y$ are isomorphisms in $\mathcal{E}$.

Ex. 3. Let $\mathcal{E}$ and $\mathcal{E}'$ two exact categories, and let $F\mathcal{E} \to \mathcal{E}'$ an exact functor that is an equivalence between them. Show that $K_n(\mathcal{E}) \simeq K_n(\mathcal{E}')$ for all $n \geq 0$.

Bonus If you are interested in these topics, I encourage you to read about Quillen's Theorem A and Theorem B. They require a bit more background so we won't see them, but they are at the base of the proof of several of the fundamental theorems of K-theory. And are a cool and important tool in general too!