Introduction to algebraic K-theory
Problem set 4

Ex. 1. Let R be a ring, let us denote by $Proj(R)$ the category of finitely generated projective R-modules, and by $Ch^b(R)$ the category of bounded chain complexes of R-modules. seen as exact categories in the canonical way.

Prove that

$$K_0(Proj(R)) \cong K_0(Ch^b(R)).^1$$

Hint: build a morphism in each direction, and prove they are inverses. There is a clear map induced by the inclusion $Proj(R) \to Ch^b(R)$, for the inverse consider the Euler characteristic.

Ex. 2. Let $\mathcal{C}$ be a Waldhausen category. We mentioned in class that the category $S_n\mathcal{C}$ admits a Waldhausen structure whose weak equivalences are levelwise and cofibrations are *Reedy* cofibrations.

Identify what goes wrong if we try taking cofibrations levelwise as well.

Ex. 3. Let $\mathcal{C}$ be a Waldhausen category, and let's write $K_0(\mathcal{C})$ for its K_0-group defined Grothendieck-style. Prove that

$$\pi_0(\Omega|NwS_\bullet\mathcal{C}|) \cong K_0(\mathcal{C})$$

[1] This isomorphism is true for higher K-groups (but much harder to prove) and it's called Gillet-Waldhausen theorem.