ATLANTIC TQFT SPRING SCHOOL 2026
Stable Homotopy and Unitary TQFTs

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Contents

Preface

These notes were taken by Skyler Lee during the course *Stable Homotopy and Unitary TQFTs* by Luuk Stehouwer, as part of the 2026 Atlantic TQFT Spring School, which took place at Dalhousie University from May 18-22, 2026.[1]

[1]All errors are due to the note-taker. If you find any mathematical errors or typesetting mistakes, please contact `skylerstlee@gmail.com`.

1. Lecture 1

1.1. **Overview of the course.** The main goal of this course is to carefully define nonextended TQFTs. The side goals include

(i) Discuss invertible TQFTs and stable homotopy theory.

(ii) Provide remarks on the (∞, n)-version.

We will mainly be working with classical notions of TQFTs, due to Atiyah, and inside 1-categories. However, we will seek descriptions that generalize quite well to ∞- and (∞, n)-category variants.

1.2. **Topological quantum field theories.** As given in the name, we will need to develop quite bit of topological machinery and some aspects of quantum field theory to carefully define TQFTs.

"Quantum field theory". An n-dimensional quantum field theory Q (where n is spacetime) "should" assign

(i) To every $(n-1)$-dimensional manifold Y^{n-1}, a complex Hilbert space $Q(Y)$ of states

(ii) To every spacetime $M : Y_1 \to Y_2$, interpreted as a bordism between Y_1, Y_2, a linear map
$$Q(M) : Q(Y_1) \to Q(Y_2)$$
which should be interpreted as a "time-evolution operator".

Our goal will be to make a (topological) version this definition rigorous.

Definition 1.2.1. A *(co)bordism* from closed manifolds Y_1^{n-1} to Y_2^{n-1} is

(i) A compact manifold M

(ii) A partition $\partial M \equiv \partial_{in} M \amalg \partial_{out} M$

(iii) Diffeomorphisms $Y_1 \cong \partial_{in} M$ and $Y_2 \cong \partial_{out} M$

The "arrow of time" is important for geometric structures. Given a compact manifold M, we get the short exact sequence of vector bundles
$$0 \to T\partial M \to TM|_{\partial M} \to \text{normal}(\partial M \hookrightarrow M) \to 0$$
where $\text{normal}(\partial M \hookrightarrow M) \cong \mathbb{R}$ using the convention of an "outgoing" normal vector. This is meaningful insofar as M has a metric (contractible datum). The other trivialization (based on the opposite choice of outgoing normal) leads to changes by $-id_{\mathbb{R}}$.

1.3. **Tangential structures.** The rank r vector bundles $V \to X$ with metric are "classified by"
$$X \to BO(r) \equiv Gr(\mathbb{R}^r \subset \mathbb{R}^\infty)$$
where $Gr(\mathbb{R}^r \subset \mathbb{R}^\infty)$ is the Grassmannian of r-dimensional linear subspaces of $\mathbb{R}^\infty$.

The vector bundle isomorphism
$$\begin{array}{ccc} V_1 & \xrightarrow{\cong} & V_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & x_2 \end{array}$$

yields a homotopy

$$\begin{array}{ccc} X_1 & \longrightarrow & \\ \downarrow & \Downarrow & BO(r) \\ X_2 & \longrightarrow & \end{array}$$

Definition 1.3.1. A $(X \xrightarrow{\rho} BO(n))$ *tangential structure* on M is a map $M \to X$ and a homotopy

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow \rho \\ M & \xrightarrow[TM]{} & BO(n) \end{array}$$

Remark 1.3.1. A map of tangential structures involve tetrahedra, namely with the same $BO(n)$ at one vertex, and 2 pairs of vertices $M_1 \to X_1$ and $M_2 \to X_2$.

Fact. There exist maps

$$BO \equiv \operatorname{colim}(BO(n) \xrightarrow{-\oplus\mathbb{R}} BO(n+1) \to \cdots)$$

Moreover, if one considers the direct product

$$BO(n) \times BO(m) \xrightarrow{\oplus} BO(n+m)$$

then this can also be considered in the colimit. That is, $(BO, \oplus)$ is an $\mathbb{E}_\infty$-group. Moreover, it has $\ominus$, which should be considered an "orthonormal complement".

Definition 1.3.2. A *stable tangential structure* is a map $X \to BO$ along with the homotopy pullback

$$\begin{array}{ccc} X(n) & \dashrightarrow & BO(n) \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & BO \end{array}$$

Lemma 1. An $X(n+k)$-structure on $TM \oplus \mathbb{B}^k$ is equivalent to an $X(n)$-structure on TM.

Example 1.3.1. (i) If $X = BO$, then this yields no data.

(ii) Define

$$X = BSO \equiv \pi_{\geq 2} BO$$

That is, removing information from $\pi_1 BO$,

$$\begin{array}{ccccc} & & BSO & & \\ & \nearrow & \downarrow & & \\ M & \longrightarrow & BO & \xrightarrow{det = w_1} & B\mathbb{Z}/2 \end{array}$$

yields $\pi_1(M) \to \mathbb{Z}/2$.

(iii) $X = BSpin \equiv \pi_{\geq 3} BO$ setting $\pi_2 BO = \mathbb{Z}/2$. A spin structure also requires the composition

$$M \xrightarrow{TM} BO \xrightarrow{w_2} B^2\mathbb{Z}/2$$

to vanish.

(iv) $X = \{pt\}$ yields framing.

(v) $X = BO \times BG \xrightarrow{pr_1} BO$, that is, an X-structure is a G-local system.

Remark 1.3.2. Note that $BSpin(n) = \pi_{\geq 3}BO(n)$ only if $n \geq 3$, so the construction above must be done in the prescribed order.

Definition 1.3.3. The *X-structured bordism category $Bord^X_{n,n-1}$* has as objects Y^{n-1} with $X(n-1)$-structure, and morphisms $Y_1 \to Y_2$ are bordisms M with $X(n)$-structure, with data stating that

$$TY_2 \oplus \mathbb{R} \cong T\partial_{out}M \oplus \mathbb{R} \cong TM|_{\partial_{out}M}$$

preserves $X(n)$-structures, and similarly

$$TY_1 \oplus \mathbb{R} \cong T\partial_{in}M \oplus \mathbb{R} \cong_{id \oplus -id_{\mathbb{R}}} T\partial_{in}M \oplus \mathbb{R} \cong TM|_{\partial_{in}M}$$

preserves $X(n)$-structures.[2] Composition is defined by gluing, and the identity is given by $Y \times [0,1]$.

Now we can finally define TQFTs with some clarity and care.

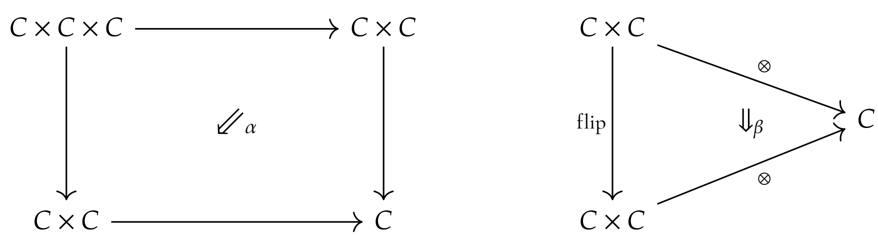
Definition 1.3.4. A *topological quantum field theory (TQFT)* is a symmetric monoidal functor

$$Z : Bord^X_{n,n-1} \to C$$

by which we have

$$Z(Y_1 \amalg Y_2) \cong Z(Y_1) \otimes Z(Y_2)$$

Recall that an $\mathbb{E}_\infty$-category (i.e., a symmetric monoidal category) is an $\mathbb{E}_\infty$-monoid in **Cat**, i.e., we have the tensor operation $C \times C \xrightarrow{\otimes} C$ that is associative and equipped with the braiding,



Example 1.3.2. **($Bord^X_{n,m}, \amalg$).** Consider the inclusion

$$(Vect_k, \otimes_k) \hookrightarrow (sVect_k, \hat{\otimes}_k)$$

where the inclusion is into the purely even degree sector of $sVect_k$, which are super vector spaces.[3]

Definition 1.3.5. An *X-structured TQFT* is an $\mathbb{E}_\infty$-functor

$$Bord^X_{n,n-1} \to sVect_{\mathbb{C}}$$

A TQFT is *bosonic* if it lands in $Vect_{\mathbb{C}}$ (namely, inclusion as purely even degree inside $sVect_{\mathbb{C}}$).

[2]This is all defined up to $X(n)$-preserving diffeomorphisms, relative to ∂.

[3]Super vector spaces are vector spaces V such that

$$V = V_0 \oplus V_1$$

where V_0 consists of purely even degree, and V_1 consists of purely odd degree. Moreover, $\hat{\oplus}$ uses $\mathbb{Z}/2$ multiplication, namely

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$$

where $|v| = 0, 1$ based on the even/odd degree of $v \in V$.

Remark 1.3.3. Let $X = BO \times BG$. If Y^{n-1} has a trivial local system, then there exists X-structured automorphism g of M for all $g \in G$. This translates to a representation of G on $Z(Y)$.

2. Lecture 2

2.1. Unitary TQFTs.

Yesterday, we defined a TQFT as the map

$$Y^{k-1} \mapsto Z(Y) \in Hilb_{\mathbb{C}}$$

A pressing question is then: what should a unitary TQFT be? We could instead define a TQFT as the map

$$Bord^X_{n,n-1} \to (sHilb_{\mathbb{C}}, \otimes)$$

Now we have to actually define what we mean by this Hilbert space category. The objects are super Hilbert spaces. However, what are the morphisms?

- We could consider all linear maps. But the forgetful functor from Hilbert spaces to vector spaces is an equivalence, so this does not seem like the proper choice.
- An alternative is to consider unitary morphisms (isometries), namely such that $T^\dagger = T^{-1}$. Unfortunately, this also does not work well.

To make headway towards this, we introduce the following definitions.

Definition 2.1.1. *(Selinger)* A *†-category* $(C, \dagger)$ is a category C equipped with a †-structure,

$$(-)^\dagger : C \to C^{\text{op}}$$

such that $(-)^{\dagger\dagger} = id_C$, and $x^\dagger = x$ for all $x \in C$.

Definition 2.1.2. A *†-functor*

$$F : (C, \dagger) \to (D, \dagger)$$

preserves †-structure strictly.

A *unitary natural isomorphism* satisfies

$$\phi_x^\dagger = \phi_x^{-1}$$

on all components. These form $Cat_1^\dagger$, a $(2,1)$-category.

An $\mathbb{E}_\infty$ *†-category* is an $\mathbb{E}_\infty$-monoid in $Cat_1^\dagger$. (That is, $\otimes : C \times C \to C$ is a †-functor and $\alpha, \beta, \ldots$ are unitary.)

Example 2.1.1. Consider *Hilb*. Define a †-structure on $Bord^X_{n,n-1}$, and require that

$$Bord^X_{n,n-1} \to sHilb$$

is an $\mathbb{E}_\infty$ and †-functor.

Example 2.1.2. If $X = BO$, then obviously $Bord^{BO}_{n,n-}$ is a †-category.

If $X = BSO$, this makes intuitive sense.

(i) Reverse the bordism from Y_1 to Y_2, i.e.,

$$Y_2^\vee \xrightarrow{M^r} Y_1^\vee$$

where $Y^\vee = Y$ but with $X(n)$-structure on $TY \oplus \mathbb{R}$ in the original but composed with $id_{TY} \oplus -id_{\mathbb{R}}$.

(ii) Now orientation reverse the whole of M.

Exercise 1. Show that if $X = * \to BO(n)$, then in general, $Bord^X_{n,n-1}$ does not admit a †-structure.

Importantly, we have a natural correspondence between

$$X(n)\text{-structure on } TY \oplus \mathbb{R} \Leftrightarrow X(n-1)\text{-structure on } TY$$

2.2. **Duality.**

Definition 2.2.1. Let C be an $\mathbb{E}_\infty$ category and $x \in C$. Then, a *dual* of x is $x^\vee \in C$ together with evaluation and coevaluation maps,

$$ev_x : x^\vee \otimes x \to 1$$

$$coev_x : 1 \to x \otimes x^\vee$$

satisfying the identities

$$\begin{array}{ccccc} x & \longrightarrow & x \otimes x^\vee \otimes x & \xrightarrow{id \otimes ev} & x \\ & \searrow^{\cong} & \nearrow_{coev \otimes id} & & \\ & & 1 \otimes x & & \end{array}$$

with the other diagrams being the identity.

Fact. Duals are unique. If $(x', ev', coev')$ is another dual to x, then there eixsts a unique map

$$x^\vee \to x'$$

intertwining *ev* maps. In particular, if $F : C \to D$ is $\mathbb{E}_\infty$, then

$$F(x^\vee \otimes x) \cong F(x^\vee) \otimes F(x)$$

since

$$F(x^\vee \otimes x) \xrightarrow{F(ev)} F(1_C) \cong 1_D$$

And therefore $F(x^\vee) \cong F(x)^\vee$.

Example 2.2.1. If Z is a TQFT, then it lands in a dualizable vector space.

Example 2.2.2. $Bord^X_{n,n-1}$ has duals.

$$Y^\vee \amalg Y \xrightarrow{ev} \varnothing$$

is simply $Y \times [0,1]$, due to the ingoing/outgoing orientations.

Definition 2.2.2. Given duals for all $x \in C$, define the *dual functor*, an $\mathbb{E}_\infty$-functor

$$C \xrightarrow{(-)^\vee} C^{op}$$

defined on $f : x \to y$ by the pullback

$$\begin{array}{ccc} y^\vee \otimes x & \xrightarrow{f^\vee \otimes id_x} & x^\vee \otimes x \\ {\scriptstyle id_y \otimes f}\downarrow & & \downarrow{\scriptstyle ev_x} \\ y^\vee \otimes y & \xrightarrow[ev_y]{} & 1 \end{array}$$

Moreover, the double dual functor satisfies $(-)^{\vee\vee}_C \cong id_C$, also referred to as the *pivotal structure*.

Now, we would like a map like $V \to \overline{V}^\vee$, where $\overline{V}^\vee$ is given a Hilbert space structure, such that the functor $\overline{\cdot}^\vee$ is an identity on objects in the category.

2.3. **Hermitian construction.** Let C be an $\mathbb{E}_\infty$-category with all duals chosen and a $\mathbb{E}_\infty$ $\mathbb{Z}/2$-action[4]

$$\bar{\cdot} : C \to C$$

That is, an $\mathbb{E}_\infty$ functor $\bar{\cdot} : C \to C$ and equivalence $\bar{\bar{\cdot}} \cong_\eta id_C$, with the condition

$$\eta_{\bar{x}} = \overline{\eta_x}$$

Then, $(-)^\vee$ induces a $\mathbb{Z}/2$-action on $C^{\cong}$. Namely,

$$C^{\cong} \xrightarrow{(-)^\vee} (C^{op})^{\cong} \xrightarrow{(-)^{-1}} C^{\cong}$$

This $\mathbb{Z}_2$-action *commutes*[5] with the $\mathbb{Z}/2$-action $\bar{\cdot}$, since the latter is $\mathbb{E}_\infty$,

$$F : \mathbb{E}_\infty \Rightarrow F \cdot (-)^\vee \cong (-)^\vee \cdot F$$

and

$$\overline{(-)}^\vee \cong \overline{(-)^\vee}$$

where, on the right-hand side, $(-)^\vee$ is conducted first, and vice-versa on the left.

Definition 2.3.1. A *Hermitian pairing* on x is a $\mathbb{Z}/2$-*fixed point* for the diagonal action, namely that $\bar{x}^\vee \cong_h x$ with a condition.

Equivalently, $h : \bar{x} \otimes x \to 1$ satisfies the diagram

$$\begin{array}{ccccc}
\overline{\bar{x} \otimes x} & \xrightarrow{\bar{h}} & \bar{1} & \xrightarrow{\cong} & 1 \\
\cong \downarrow & & & & \uparrow \\
\bar{\bar{x}} \otimes \bar{x} & \xrightarrow[\cong]{} & x \otimes \bar{x} & \xrightarrow{\cong} & \bar{x} \otimes x
\end{array}$$

This formalizes the idea that $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

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[4]A $\mathbb{Z}/2$-action is a map

$$\mathbb{Z}/2 \to Aut_{Cat_1^{\mathbb{E}_\infty}}(C)$$

where the right-hand side is a 2-group, as the ambient category is a $(2,1)$-category, i.e.

$$\mathbb{Z}/2 \Leftrightarrow B\mathbb{Z}_2 \to Cat_1^{\mathbb{E}_\infty}$$

such that $* \mapsto C$.

[5]More precisely, there exists a joint $\mathbb{Z}/2 \times \mathbb{Z}/2$-action.

3. Lecture 3

The main subject of today is a rather technical intermezzo, regarding $Bord^X_{n,n-1}$ as a †-category.

3.1. Review.

Yesterday, we stated that we need a $\mathbb{E}_\infty$ $\mathbb{Z}/2$-action $\overline{\cdot}$ on $Bord^X_{n,n-1}$, such that $\overline{\cdot}^\vee$ is trivial on the objects of $(Bord^X_{n,n-1})^\cong$.

The initial idea is that $\overline{\cdot}$ is essentially "orientation reversal". This is almost correct (and correct in the oriented case), but not quite.

A more sophisticated idea is that $\overline{\cdot} \approx [(\cdot)^\vee]^{-1}$.

Remark 3.1.1. Since $\mathbb{Z}/2$ is abelian, its delooping $B\mathbb{Z}/2$ is also abelian and has group operations (i.e., inverses). If $B\mathbb{Z}/2 \to C$ is a $\mathbb{Z}/2$-action on $* \mapsto * \in C$, then one obtains an induced $\mathbb{Z}/2 \times \mathbb{Z}/2$-action

$$B\mathbb{Z}/2 \times B\mathbb{Z}/2 \xrightarrow{+} B\mathbb{Z}/2 \to C$$

In particular, both generators act in the same way and commute. Moreover, we have the $(-)$ of an action.

For any abelian A and any A-action, there exists an $A \times A$-action and an A-action (inverse). Moreover, if $A = \mathbb{Z}/2$, then the inverse is the original action.

It seems that trying to find trivializations $\bar{x}^\vee \cong x$ should be the same as $\bar{x} \cong x^\vee$. However, this is not correct.[6]

Thus, our goals for this lecture are

(i) Understand G-actions on $Bord^X_{n,n-1}$

(ii) Understand $\mathbb{Z}/2$-action on $(Bord^X_{n,n-1})^\cong$

(iii) Understand how these two actions commute

(iv) Find the $\mathbb{Z}/2$-action $\overline{\cdot}$ such that $\overline{Y}^\vee \cong Y$ and the construction from yesterday yields a †-structure for all $Y \in Bord^X_{n,n-1}$

3.2. G-actions on $Bord^X_{n,n-1}$.

The assignment

$$(X \to BO) \mapsto Bord^X_{n,n-1}$$

is functorial in the slice category over BO. Thus, we may as well try to understand G-actions on $X \xrightarrow{\rho} BO$.

Consider the following (un)straightening in topology, given by the diagram

$$\begin{array}{ccccccc}
 & & \alpha(*) & = & F & \xmapsto{\text{fiber transport}} & \\
 & & & & \downarrow & & \\
\operatorname{colim}_{BG} \alpha & = & E = X & \longrightarrow & BG & \xrightarrow{\alpha} & \varsigma \\
\downarrow & & \downarrow & & & & \\
\operatorname{colim}_{BG} * & = & BG = BO & \longleftarrow\!\!| & & &
\end{array}$$

where $\varsigma \equiv Gpd_\infty$, and the map α is given by $* \mapsto \operatorname{fib} \rho \in \varsigma$.

[6] As we will see, we need to expand our group action to $O(n)$ rather than $\mathbb{Z}/2$.

Since $X \xrightarrow{\rho} BO$ is already equivalent to the action of O in the fibers of ρ, the G-action on $X \to BO$ is equivalent to the action of $G \times O$ on the fibers of ρ. That is, it corresponds to the pullback diagram

$$\begin{array}{ccc} & & \operatorname{fib}\rho \\ & & \downarrow \\ X & \xrightarrow{i} & \hat{X} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\hat{\rho},p)} \\ BO & \longrightarrow & BO \times BG \end{array}$$

This is equivalent to

$$\begin{array}{ccc} X & & \\ {\scriptstyle i}\downarrow & \overset{\rho}{\searrow} & \\ \hat{X} & \xrightarrow{\hat{\rho}} & BO \\ {\scriptstyle p}\downarrow & & \\ BG & & \end{array}$$

In general, functoriality in G is given by

$$\begin{array}{ccc} BG_1 & \overset{\alpha_1}{\searrow} & \\ \downarrow & \Downarrow & \varsigma \\ BG_2 & \underset{\alpha_2}{\nearrow} & \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} \alpha_1(*) & \xrightarrow{\cong} & \alpha_2(*) \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & E_2 \\ \downarrow & \lrcorner & \downarrow \\ BG_1 & \longrightarrow & BG_2 \end{array}$$

and specializing to $G_1 = O, G_2 = O \times G$ yields the original pullback diagram.

Example 3.2.1. A familiar example of the above construction is the pullback diagram of $X(n) \to X$,

$$\begin{array}{ccc} X(n) & \xrightarrow{i} & X \\ \downarrow & \lrcorner & \downarrow \\ BO(n) & \longrightarrow & BO \end{array}$$

If A is abelian, define actions via

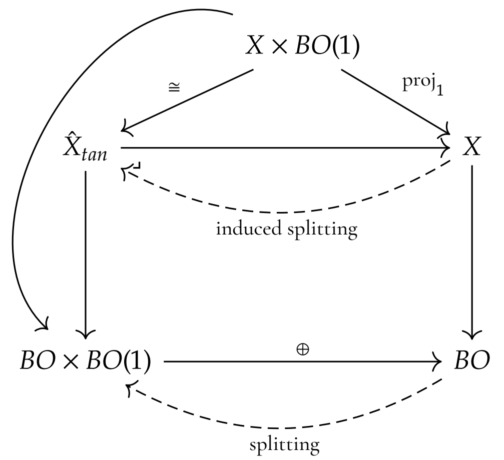
$$\begin{array}{c} F \\ \downarrow \\ E \\ \downarrow \\ BA \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} F & \xrightarrow{=} & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \\ BA \times BA & \xrightarrow{+} & BA \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{E} & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \\ BA & \xrightarrow{-} & BA \end{array}$$

where the second diagram defines the diagonal action and the third is the inverse action.

3.3. **$\mathbb{Z}/2$-action on ($Bord^X_{n,n-1}$).** Consider the $O \times O(1)$-action on X given by restricting the O-action along

$$BO \times BO(1) \hookrightarrow BO \times BO \xrightarrow{\oplus} BO$$

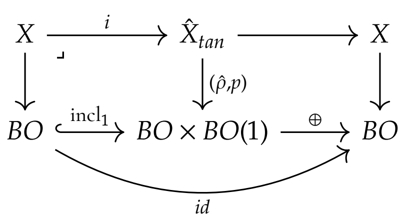
inside the diagram



where the long left-hand map $X \times BO(1) \to BO \times BO(1)$ is given by the matrix

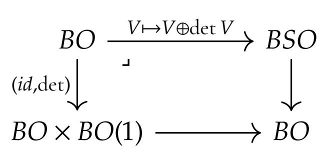
$$\begin{pmatrix} \rho & \text{incl} \\ 0 & id_{BO} \end{pmatrix}$$

The splittings yield the second pullback



Remark 3.3.1. A ($\hat{X}_{tan} \xrightarrow{\hat{\rho}} BO$) *tan*-structure on $V \to Y$ is a line bundle $L \to Y$ and X-structure on $L \oplus V$. Thus, the $\hat{X}_{tan}$-structure induced by the X-structure on $V \to Y$ is the corresponding X-structure on $V \oplus \mathbb{R}$. Concretely, this means that the $\mathbb{Z}/2$-action on X-structures is essentially "stabilize once, then reflect". In conclusion, $\hat{X}_{tan}$ gives a $\mathbb{Z}/2$-action on all $Bord^X_{n,n-1}$ that agrees with $Y^\vee$ on objects.

Example 3.3.1. Let $X = BSO$. Then,



The $\hat{X}_{tan}$-structure on $V \to Y$ is a line bundle $L \to Y$, and the orientation on $V \oplus L$ (i.e., $L \cong \det V$).

Example 3.3.2. Let $X = BSpin$. Then,

$$\begin{array}{ccc} BPin^- & \longrightarrow & BSpin \\ \downarrow & \lrcorner & \downarrow \\ BO \times BO(1) & \longrightarrow & BO \end{array}$$

Remark 3.3.2. Often, there exist $H(n) \to O(n)$ representations inducing $X(n) = BH(n) \to BO(n)$. In this case, we can use short exact sequences instead to express the $\mathbb{Z}/2$-actions, and we can work entirely at the Lie group level.

$$\begin{array}{ccc} BSO(n) & \xrightarrow{r_v(-)r_v^{-1}} & BSO(n) \\ \downarrow & \lrcorner & \downarrow \\ BO(n) & \underset{=}{\overset{r_v(-)r_v^{-1}}{\underset{\Downarrow}{\longrightarrow}}} & BO(n) \end{array}$$

3.4. **Commutation of the two actions.** Let $\hat{X} \to BO \times BG$ be a G-action on $X \to BO$. Then, the data that the G-action commutes with duals is exactly encoded by the $G \times O(1)$-action given by restricting the $G \times O$-action along the same map as above.

Example 3.4.1. The way that the $\mathbb{Z}/2$-action $\hat{X}_{tan}$ commutes with itself is governed by

$$BO(1) \times BO(1) \to BO \times BO \xrightarrow{\oplus} BO$$

The key point is that the sum in $BO \times BO$ is *different* than the sum in $BO(1) \times BO(1)$. Indeed, the diagonal $\overline{\cdot}^{\vee}$ in here is *not* null-homotopic![7]

In conclusion, for $\overline{\cdot}$ use

$$\begin{array}{ccc} \hat{X}_{nor} & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ BO \times BO(1) & \xrightarrow{\oplus} & BO \end{array}$$

to yield a †-structure on $Bord^X_{n,n-1}$.

[7]To see this, consider the action of O on a 1-category C, i.e.

$$\pi_{\leq 1} O = \mathbb{Z}/2 \times B\mathbb{Z}/2$$

where the first component corresponds to $\overline{\cdot}$ while the second is the choice of involutive natural automorphis of id_C.

The group operation on BO induces

$$(w_1, w_2)(w_1', w_2') = (w_1 + w_1', w_2 + w_2' + w_1 w_1')$$
$$\Rightarrow (w_1, 0)(w_1', 0) = (w_1 + w_1', w_1 w_1')$$

4. Lecture 4

Today's subject will be invertible TQFTs. The upshot will be that invertible TQFTs can be computed using stable homotopy theory.

Definition 4.0.1. A TQFT $Bord^X_{n,n-1} \to sVect$ is *invertible* if it lands in $Pic(sVect)$.[8]

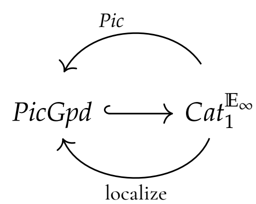
Definition 4.0.2. A *Picard groupoid* $\mathcal{G}$ is an $\mathbb{E}_\infty$-category such that all morphisms and objects are invertible under the associated tensor product and composition. That is, for all $x \in \mathcal{G}$, there exists $x^{-1} \in \mathcal{G}$ such that $x \otimes x^{-1} = x^{-1} \otimes x = 1$.

Thus, the TQFT's values on $(n-1)$-dimensional manifolds are invertible vector spaces.

Remark 4.0.1.
- TQFTs can be used to study topological order. Low-energy gapped quantum systems should lead to a TQFT.
- In this sense, an invertible TQFT corresponds to an invertible phase, also referred to as a symmetry-protected topological phase.
- $(n+1)$-dimensional invertible TQFTs classify anomalies of n-dimensional TQFTs.

Example 4.0.1. $Pic(Vect)$ has 1-dimensional vector spaces as objects, and scalar multiplications as morphisms.

Fact. From inclusion of the $(2,1)$-category $PicGpd$ into $Cat_1^{\mathbb{E}_\infty}$, we get a right adjoint from Pic, and a left adjoint from localizing by adding inverses.



This yields

$$Fun_{\mathbb{E}_\infty}(Bord^X_{n,n-1}, Pic(C)) \cong Fun_{PicGpd}(LBord^X_{n,n-1}, Pic(C))$$

so we can try to understand functors in the first category by examining the latter.

Remark 4.0.2. Note that an $\mathbb{E}_\infty$ groupoid need not be Picard, as the objects may not be invertible under tensor products.

4.1. Homotopy hypothesis.

Homotopy hypothesis. 1-groupoids correspond to 1-types X

$$\text{1-groupoid } \mathcal{G} \Leftrightarrow \text{1-type } X$$

where X is classified, up to equivalence, by $\pi_0(X)$ (set of isomorphism classes of objects), and for all $x \in \pi_0(X)$, $\pi_1(X,x) = \mathrm{Aut}_{\mathcal{G}}(x)$.

Picard groupoids then correspond to "stable" 1-types inside Sp, spectra.

$$Pic \text{ groupoid} \Leftrightarrow \text{"stable" 1-types} \subset Sp$$

Theorem 1. (Hoang) *PicGpd* are classified by

(i) π_0, an abelian group
(ii) $\pi_1 = \pi_1(X,1) = Aut_{\mathcal{G}}(1)$ where 1 is the monoidal unit, which is also an abelian group

[8] $Pic(C)$ is the maximal Picard groupoid contained inside C.

(iii) K-invariant given by tr $\beta_{x,x} \in Aut_{\mathcal{G}}(1)$ where $\beta_{x,x}$ is the braiding of x, and the map

$$\pi_0/2 \to \pi_1$$

is given by the homomorphism $x \mapsto \text{tr } \beta_{x,x}$.

Example 4.1.1. For $Pic(Vect_k)$ above, $\pi_0 = 1$, $\pi_1 = k^X$, and $K = 0$.

Example 4.1.2. For $Pic(sVect_{\mathbb{C}})$, $\pi_0 = \mathbb{Z}/2$ (for odd and even lines); $\pi_1 = \mathbb{C}^X$. Now

$$K : \mathbb{Z}/2 \hookrightarrow \mathbb{C}^X$$

is given by the map

$$v \otimes v \mapsto_{\beta_{odd,odd}} -v \otimes v$$

which is nontrivial due to the braiding (else it would be the zero map).

4.2. **General techniques to compute localizations.** Consider a category C,

$$\pi_0 LC \cong \text{obj}(C)/(x \sim y)$$

given by the objects of C modded by the equivalence relation generated by morphisms $x \to y$.

Definition 4.2.1. C is *reversible* if $Hom_C(x, y) \neq \emptyset$ implies $Hom_C(y, x) \neq \emptyset$.

Theorem 2. **(Bokstedt-Svane)** If C is reversible, then

$$GroupComplete(End_C(1)) \to End_{LC}(1) = \pi_1(LC)$$

is not an isomorphism, but is surjective with kernel generated by

$$(1 \xrightarrow{f_1} x \xrightarrow{f_2} 1 \xrightarrow{g_1} x \xrightarrow{g_2} 1) \sim (1 \xrightarrow{f_1} x \xrightarrow{g_2} 1 \xrightarrow{g_1} x \xrightarrow{f_2} 1)$$

Remark 4.2.1. $LBord^X_{n,n-1}$ can be computed using fancier techniques like spectral sequences.

$$\pi_0 LBord^X_{n,n-1} = \Omega^{X(n)}_{n-1}$$

is called the bordism group for structure $X(n)$

$$(X(n-1), \text{closed manifolds with } \amalg)/X(n)\text{-bordisms}$$

Meanwhile,

$$\pi_1 LBord^X_{n,n-1} = \Omega^{X(n)}_n$$

is the Reinhart unstable bordism group; that is, the bordism group for the stable tangential structure $X(n) \to BO(n) \to BO$, given by the pullback along the diagram

$$\begin{array}{ccccc} Y_1 & & \dashrightarrow & & BO(n) \\ \downarrow & & & & \downarrow \\ X(n) & \longrightarrow & BO(n) & \longrightarrow & BO \end{array}$$

leading to the unstable nomenclature. Finally, the K-invariant is

$$\Omega^{X(n)}_{n-1} \to \Omega^{X(n)}_n$$
$$Y \mapsto Y \times S^1$$

Now we are ready to present the analogous theorem to the above for morphisms.

Theorem 3. For domain category C and target D, there exists a short exact sequence

$$0 \to Ext(\pi_0(C), \pi_1(D)) \to Fun_{PicGpd}(C, D)/\cong \to \{f_0 : \pi_0(C) \to \pi_0(D), f_1 : \pi_1(C) \to \pi_1(D)\} \to 0$$

where f_0, f_1 are such that they are compatible with the K-invariant

$$K_D \circ f_0 = f_1 \circ K_C$$

Example 4.2.1. Returning to our example of $D = sVect_{\mathbb{C}}$, $\pi_0 = \mathbb{Z}/2, \pi_1 = \mathbb{C}^X$. Thus,

$$Ext(\pi_0(C), \mathbb{C}^X) = 0$$

so we can ignore this term in the sequence and the latter map in the sequence is an isomorphism. Thus, the computation is given by the diagram

$$\begin{array}{ccc} \mathbb{C}^X & \longleftarrow & \pi_1(C) \\ {\scriptstyle K_{Pic}(sVect)} \uparrow & & \uparrow {\scriptstyle K_C} \\ \mathbb{Z}/2 & \longleftarrow & \pi_0(C) \end{array}$$

where only the upper map needs to be given, as by the injectivity of the left-hand map we can define the bottom map from $\pi_0(C)$.

Theorem 4. X-structured invertible TQFTs are precisely $Z \in Hom(\Omega_n^{X(n)}, \mathbb{C}^X)$.

Example 4.2.2. Bosonic X-structured invertible TQFTs are those Z such that

$$Z(Y \times S^1) = 1$$

for all $Y \in Bord^X_{n,n-1}$.

Example 4.2.3. $\Omega_2^{SO(2)} \cong \mathbb{Z}$ is unstable, given by the map

$$[\Sigma] \mapsto \chi[\Sigma]/2$$

Then, oriented 2-dimensional invertible TQFTs are

$$\begin{aligned} Hom(\Omega_2^{SO(2)}, \mathbb{C}^X) &\cong \mathbb{C}^X \\ Z &\mapsto Z(S^2) \end{aligned}$$

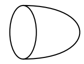
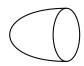
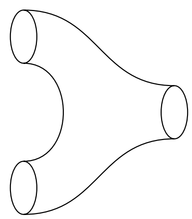
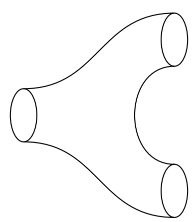
4.3. **Unitary TQFTs.** [9]

Fact. 2-dimensional bosonic oriented TQFTs correspond to commutative Frobenius algebras. The subset of 2-dimensional unitary TQFTs then correspond to †-Frobenius algebras. That is,

$$\begin{array}{ccc} 2d \text{ oriented TQFTs} & \Longleftrightarrow & \text{commutative Frobenius algebras} \\ \cup & & \\ 2d \text{ unitary TQFTs} & \Longleftrightarrow & \dagger\text{-Frobenius algebras} \end{array}$$

The correspondence is given by the dictionary

[9] We skipped this yesterday, and are returning to provide some more details and examples.

 $S^1 \to \varnothing$	$Z(S^1) = A \xrightarrow{\lambda} \mathbb{C}$
 $\varnothing \to S^1$	unit
 $S^1 \sqcup S^1 \to S^1$	multiplication $A \otimes A \xrightarrow{\mu} A$
 $S^1 \to S^1 \sqcup S^1$	$A \xrightarrow{\Delta} A \otimes A$

Theorem 5. 2-dimensional bosonic unitary oriented TQFTs are commutative Frobenius algebras equipped with a Hilbert space structure such that

$$\mu^\dagger = \Delta$$

and

$$\lambda^\dagger = (\mathbb{C} \xrightarrow{unit} A)$$

Theorem 6. 1-dimensional unitary TQFTs are O-fixed points for $\pi_{\leq 1}X$ given by

$$\begin{array}{ccc} \pi_{\leq 1}X & & \\ \downarrow & & \\ \pi_{\leq 1}O & \xrightarrow{\ominus} & \pi_{\leq}O = \mathbb{Z}/2 \times B\mathbb{Z}/2 \end{array}$$

where $\mathbb{Z}/2 \times B/ZZ/2$ acts via $(\bar{\cdot}, (-1)F)$ on $(sHilb)^{\cong,\mu}$.

That is, for all $x \in X$, X_x is a Hilbert space with symmetries given by a unitary group action of $\pi_1(X, x)$ on X_x.

5. Lecture 5

Today's lecture focuses on adding unitarity to invertible TQFTs, and outlining what happens when we go to higher categories.

5.1. **Unitary invertible TQFTs.**

Definition 5.1.1. A †-*Picard groupoid* is an $\mathbb{E}_\infty$ †-category in which all morphisms and objects are invertible.

Theorem 7. (Kreck-Stolz-Teichner) There exists a classification of †-Picard groupoids given by

(i) $\pi_0^u \twoheadrightarrow_p \pi_0$, a surjection.[10]

(ii) Involutions of π_1 by $(-)^\dagger$.

(iii) Monomorphisms $K_\dagger : \ker p \hookrightarrow \frac{sa}{pos}$.[11]

(iv) $K : \pi_0/2 \to \pi_1^u$ where $\pi_1^u = \{f \in \pi_1 | f^\dagger f = 1\}$[12]

Example 5.1.1. Consider $sHerm = Herm(sVect_{\mathbb{C}})$; then its invertible part $Pic(sHerm)$ is a Picard groupoid. It has unitary isomorphism classes of objects $\mathbb{C}_+$ for positive definite, $\mathbb{C}_-$ for negative definite; $\mathbb{C}_+^{0/1}$, and $\mathbb{C}_-^{0/1}$. In this case,

$$\pi_0^u(Pic(sHerm)) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \twoheadrightarrow \mathbb{Z}/2 = \pi_0(Pic(sHerm))$$

and the involution of π_1 by $(-)^\dagger$ is simply the action of $\mathbb{Z}/2$. Moreover,

$$\ker p = \{\mathbb{C}_+, \mathbb{C}_-\} \xrightarrow{K_\dagger} \frac{sa}{pos} \cong \mathbb{Z}/2$$

$$sa = \mathbb{R}^X$$

$$pos = \mathbb{R}_{>0}$$

and

$$\mathbb{C}_- \xrightarrow{f} \mathbb{C}_+ \Rightarrow f^\dagger f = -1$$

Theorem 8. There exists an exact sequence

$$0 \to Ext(\pi_0(C), \pi_1^u(D)) \to Fun^\dagger_{\mathbb{E}_\infty}(C, D)/\sim_u \to \{f_0 : \pi_0^u(C) \to \pi_0^u(D), f_1 : \pi_1(C) \to \pi_1(D)\}$$

$$\to \mathrm{Hom}(\pi_0(C)/2, \pi_1^u(D)) \times \mathrm{Hom}\left(\ker p_C, \frac{sa(D)}{pos(D)}\right)$$

---

[10] π_0^u of C consists of objects $x, y \in C$ modded by the equivalence of $x \sim y$ if there exists $f : x \to y \in C$ with $f^\dagger = f^{-1}$.

[11] We also have a map

$$\ker p \xrightarrow{K_\dagger} \frac{sa}{pos}$$

given by maps $x \mapsto f^\dagger f$ where $f : 1 \to x$ is an isomorphism, and

$$sa \equiv \{f \in \pi_1 | f^\dagger = f\}$$

$$pos \equiv \{f^\dagger f | f \in \pi_1\}$$

[12] If $x \in \pi_0^u/2$, we have

$$x \otimes x^{-1} \cong 1$$

so $x \in \ker p$ and there exists a unitary

$$x \otimes x^{-1} \otimes x \otimes x^{-1} \cong_u 1$$

where f_1 is $\dagger$-equivariant such that it takes $\ker p_C$ to $\ker p_D$,

$$K_D \circ p_D f_0 = f_1 K_C \circ p_C$$

and the final arrow is given by

$$(f_0, f_1) \mapsto \left(f_0^*(K_D) - f_{1,*}(K_C), (f_0|_{\ker p})^*(K_\dagger^D) - (f_1)_*(K_\dagger^C)\right)$$

Corollary 1. *Unitary, X-structured, invertible TQFTs are classified by $\mathbb{Z}/2$-equivariant*

$$\Omega_d^{X(d)} \xrightarrow{Z} \mathbb{C}^X$$

such that for all $\varnothing \xrightarrow{M} Y^{d-1}$, $Z(M^\dagger M) > 0$.

Proof. Note that

$$\pi_1(C) = \pi_1(LBord_{d,d-1}^X) = \Omega_d^{X(d)}$$

and

$$\pi_1(Pic(sHilb)) = \mathbb{C}^X$$

so $Z = f_1$, $K_\dagger^D = 0$, and the only extra condition is that $(f_1)_*(K_\dagger^C) = 0$. Thus, we only have to check that

$$Z(M^\dagger M) = 1 \in \frac{sa(Pic(sHilb))}{pos(Pic(sHilb))}$$

which only requires $Z(M^\dagger M)$ to be a positive number. $\square$

Theorem 9. For all $M : \varnothing \to Y$ as a morphism in $Bord_{n,n-1}^{X(n)}$, $M^\dagger M$ bounds an $X(n+1)$-manifold.

Corollary 2. *If* $Z(M^\dagger M) = 1$, *then Z yields a homomorphism*

$$\Omega_d^X \to U(1)$$

5.2. **Higher categories.** While this concludes the main thrust of the lectures, many of the concepts and results generalize to higher categories.

Remark 5.2.1. There exists $Bord_n^X$ as an $\mathbb{E}_\infty$ (∞, n)-category. Rotate framing

$$BO(n) \xrightarrow{\alpha_n} Cat_{(\infty,n)}^{\mathbb{E}_\infty}$$
$$* \mapsto Bord_n^{Fr}$$

Then

$$Bord_n^X = \operatorname*{colim}_{X_n} \alpha_n \circ \rho \cong Bord_n^{Fr} // \Omega X$$

where $\rho : X_n \to BO(n)$. Moreover, α_n is compatible for different n in the sense that

$$\begin{array}{ccc} O(n) & & O(n+1) \\ \circlearrowright & & \circlearrowright \\ Bord_n^{Fr} & \longrightarrow & Bord_{n+1}^{Fr} \end{array}$$

Remark 5.2.2. Recall that

$$\begin{array}{ccc} \hat{X}_{tan} & \longrightarrow & X_n \\ \downarrow & \lrcorner & \downarrow \\ BO(n-1) \times BO(1) & \xrightarrow{\oplus} & BO(n) \end{array}$$

gave $O(1)$ as an action on $Bord^{X}_{n,n-1}$ which agreed on $(Bord^{X_n}_{n,n-1})^{\cong}$ with $(-)^{\vee}$.

This is still "true" here, in the sense that it agrees with

$$Bord^{X(n)}_n \xrightarrow{(-)^{\vee}} (Bord^{X(n)}_n)^{n-op}$$

on $(Bord^{X(n)}_n)^{\cong_n}$.

If $X \to BO$, we also have $\mathbb{Z}/2$ action on $Bord^{X(n)}_n$ by $\hat{X}$, yielding

$$Bord^{X(n-1)}_{n-1} \hookrightarrow ((Bord^{X(n)}_n)^{\cong_n})^{O(1)}$$

as a "† n-category".

Definition 5.2.1. A *unitary extended TQFT* into a † n-category C is a $\mathbb{Z}/2$-equivariant

$$Bord^{X(n)}_n \to C$$

along with

$$\begin{array}{ccc} Bord^{X(n-1)}_{n-1} & \dashrightarrow & C_n \\ \big\downarrow & \swarrow & \big\downarrow \\ ((Bord^{X(n)}_n)^{\cong_n})^{O(1)} & \longrightarrow & (C^{\cong_n})^{O(1)} \end{array}$$