# EXERCISES FOR HALIFAX LECTURES

CAMPBELL WHEELER

Contents



## 1. Physics ideas and some history

### 1.1. Alexander polynomial.

Show that:

- $\Delta_{\underbrace{O\cdots O}_{n}}(t) = \delta_{n,1}$,
- $\Delta_{3_1}(t) = t^{-1} - 1 + t$,
- $\Delta_{4_1}(t) = -t^{-1} + 3 - t$,

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Choose you favorite link $L$ and compute $\Delta_L(t)$. Look it up to check.

### 1.2. **Jones polynomial.**

Show that:

- $J_{\underbrace{O\cdots O}_{n}}(t) = (-q^{-1/2} - q^{1/2})^{n-1}$,
- $J_{3_1}(t) = q + q^3 - q^4$,
- $J_{4_1}(t) = q^{-2} - q^{-1} + 1 - q + q^2$,

Choose you favorite link $L$ and compute $J_L(q)$. Look it up to check.

### 1.3. **Mirror images.**

If $\overline{L}$ denotes the mirror of a link $L$, then $J_{\overline{L}}(q) = J_L(q^{-1})$.

Hence, deduce that $3_1 \neq \overline{3_1}$.

### 1.4. **Kauffman bracket.**

Show that:

- the Kauffman bracket satisfies the Reidemeister moves II and III,
- for oriented $L$, the association $(-q^{3/4})^{w(L)}\langle L\rangle$ satisfies all three Reidemeister moves, where $w$ is the writhe (the number of positive minus negative crossings),
- the Kauffman bracket recovers the Jones polynomial via $(-q^{3/4})^{w(L)}\langle L\rangle = J_L(q)$.

## 2. TQFTs for low dimensions

### 2.1. **Finite dimensions.**

Prove that if $Z$ is a $(n+1)$-dimensional TQFT and $\Sigma$ is an $n$-manifold that $\dim(Z(\Sigma)) < \infty$.

Moreover, show that for a diffeomorphism $f : \Sigma \to \Sigma$ and $Z(\Sigma \times_f S^1) = \mathrm{tr}(Z(f))$.

### 2.2. **Low dimensions.**

Show that:

- $(0 + 1)$-dimensional TQFTs are classified by finite dimensional vector spaces with a non-degenerate inner product form,
- $(1 + 1)$-dimensional TQFTs are classified by Frobenius algebras (look this up if you don't know the definition).

Note: for the second part the easy direction is enough, i.e., $(1 + 1)$-dimensional TQFTs define Frobenius algebras.

### 2.3. **Trivial Dijkgraaf-Witten TQFT.**

Show that for a finite group $G$:

- $Z(\Sigma) = \text{Span}\{G - covers\}$ and $Z(M; \Sigma \to \emptyset)(A) = \sum_{B|\Sigma=A} \frac{1}{|\text{Aut}(A)|}$ extends to a TQFT in any dimension,
- for $(1+1)$-dimensions the Frobenius algebra is given by the centre of the group ring center($\mathbb{C}[G]$),
- (Mednykh's formula) $|\text{Hom}(\pi_1(\Sigma_g))| = |G|^{2g-1} \sum_{\alpha=\text{irred}(G)} \frac{1}{\dim(\alpha)^{2g-2}}$,
- compute $Z(S^3)$ and $Z(S^2 \times S^1)$.

### 2.4. **Braids and Reidemeister.**

Show that $\sigma_i^{-1}\sigma_i = 1$ corresponds to RII and that $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ corresponds to RIII.

### 2.5. **Markov's theorem (the easy direction).**

Prove that, for a braid $\beta$, the closure of the following braids give the same link as the closure of $\beta$:

- $\alpha\beta\alpha^{-1}$,
- $\iota_n(\beta)\sigma_n^{\pm 1}$ where $\iota_n : B_n \to B_{n+1}$ is the map which adds a straight strand to the braid in the $n+1$ position.

## 3. Witten-Reshetikhin-Turaev invariants

### 3.1. **Quasi-triangular Hopf algebras.**

Show that for a Hopf algebra $V$ with co-product $\Delta$, if $R_0 \in V \otimes V$ such that

- $R_0 \Delta R_0^{-1} = \tau\Delta$ where $\tau(v \otimes v') = v' \otimes v$,
- $(\Delta \otimes \text{id})(R_0) = (R_0)_{13}(R_0)_{23}$,
- $(\text{id} \otimes \Delta)(R_0) = (R_0)_{13}(R_0)_{12}$,

(where if $R = \sum a \otimes b$ then $R_{13} = \sum a \otimes 1 \otimes b$ etc.), then $R = \tau(R_0)$ satisfies Yang-Baxter

$$(R \otimes \text{id})(\text{id} \otimes R)(R \otimes \text{id}) = (\text{id} \otimes R)(R \otimes \text{id})(\text{id} \otimes R).$$

Such and $R$ makes $V$ a quasi-triangular Hopf algebra.

Check this for $U_q(\mathfrak{sl}_2)$ with $R = \frac{1}{4r}\sum_{n,a,b}(-1)^n q^{n(n-3)/4-ab-bn+an-1} Y^n K^b \otimes X^n K^a$. (Check Kirby-Melvin for conventions if needed.)

### 3.2. **The trefoil.**

Compute the coloured Jones polynomials for the trefoil. Check it agrees with something like

$$\sum_{k=0}^{n-1} (q^{1-n}; q)_k (q^{1+n}; q)_k$$

up to normalisation (e.g. multiplication by $(q^{n/2}-q^{-n/2})/(q^{1/2}-q^{-1/2})$).

3.3. **Lemma.**

Show that:

- $J_{LO} = [n]J_L$ where $O$ is coloured by $n$,
- $J_{L'} = q^{(n^2-1)/4}J_L$ where $L'$ is $L$ with a twirl coming from RI and $n$ is the colour or $L$ and $L'$,
- $J_{L'} = \frac{[nk]}{[n]}J_L$ where $L'$ is $L$ with an additional loop around a strand coloured by $k$ and $L$ is coloured by $n$.

3.4. **Time to compute.** Write a program to compute

$$J_{4_1,n}(q) \;=\; \sum_{k=0}^{n-1}(-1)^k q^{-k(k+1)/2}(q^{1-n};q)_k(q^{1+n};q)_k\,.$$

Try to beat $J_{4_1,100}(q)$ in 50 seconds. (My code was one line.)

Compute $J_{4_1,n}(e^{2\pi i/n})$ for $n = 10000,\dots,10010$. Try to beat 2.5 seconds per value with 2000 digits of precision.

Try to determine the asymptotic behaviour of $J_{4_1,n}(e^{2\pi i/n})$ numerically as $n\to\infty$.

## 4. Geometry of 3-manifolds and semiclassical

4.1. **Volume of tetrahedra.** Show that $\mathrm{Vol}(\Delta_z) = \Lambda(\arg(z))+\Lambda(-\arg(1-z))+\Lambda(\arg(1-z^{-1}))$, where

$$\Lambda(x) = -\int_0^x \log|2\sin(t)|dt\,.$$

(Hint: check Thurston's book if struggling.)

4.2. **SnapPy.** Download SnapPy and play! For example, draw a knot check if it is hyperbolic and if it is compute its volume.

4.3. **Pochhammer symbol.** Show that:

- $(x;e^{\hbar})_\infty = \exp(\sum_{k=1}^{\infty}\frac{x^k}{k(q^k-1)})$,
- $(x;e^{\hbar})_\infty \sim \exp(\sum_{\ell=0}^{\infty}\frac{B_\ell}{\ell!}\mathrm{Li}_{2-\ell}(x)\hbar^{\ell-1}$.

## 5. Arithmetic aspects and Habiro rings

5.1. **Units.** Show that $q^{-1}\in\mathcal{H}_{\mathbb{Z}}$.

5.2. **Ramified extensions.** Show that $|\zeta_{p^d}-1|_p < 1$ for $\zeta_{p^d}$ a primitive $p^d$-th root of unity.

5.3. **Basic Habiro elements.** For $F(t,q)=\sum_{k=0}^{\infty}\binom{2k}{k}_q t^k$ and $f_m=F(t^{1/m},\zeta_m+(q-\zeta_m))$, compute:

- $\varphi_p(\frac{1}{\sqrt{1-4t}})$ where $\varphi_p(t)=t^p$,
- $f_1+O(q-1)^5$,
- $f_5+O(q-\zeta_5)$ and check the reexpansion of $f_1$ agrees after taking frobenius.

5.4. **Frobenius on cohomology.** Compute $\varphi_p$ acting on $\sum_{k=0}^{\infty}\binom{2k}{k}^2 t^k$.

5.5. **Admissible series.** Consider, $F(t,q)=\sum_{k=0}^{\infty}q^{3k^2}t^k$. Compute the $c_{i,j}\in\mathbb{Z}$ giving

$$F(t,q)=\prod_{j\in\mathbb{Z}}\prod_{i=1}^{\infty}(q^j t^i;q)_\infty^{c_{i,j}}\,.$$

Check numerically (i.e. to finite precision) that for fixed $i$ we have $c_{i,j}=0$ for all but finitely many $j$.

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