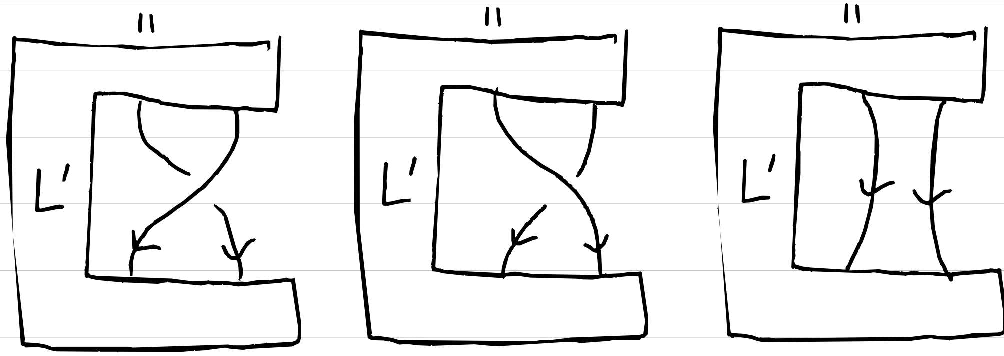
Lectures Halifax: Quantum 3-mfld. inv. 1.1

Goal: Define, compute, & study inv. of 3-mflds depending on $q = e^{\hbar}$.

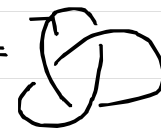
The first example of a quantum inv. is Jones' poly inv. of links. This looks very similar to Conway's desc. of Alexander's poly inv. in terms of skien relations (+ $\Delta_O(t) = 1$)

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{L_0}(t)$$

where $L_+ =$, $L_- =$, $L_0 =$



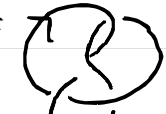
Exer: Show that

$3_1 =$ 

① $\Delta_{\overbrace{\circ\circ\cdots\circ}^{n}}(t) = \delta_{n,1}$

② $\Delta_{3_1}(t) = t - 1 + t^{-1}$

③ $\Delta_{4_1}(t) = -t + 3 - t^{-1}$

$4_1 =$ 

④ Choose your favorite

The Jones poly. sat. $J_{\circ}(q) = 1$

$$q J_{L_+}(q) - q^{-1} J_{L_-}(q) = (q^{\frac{1}{2}} - q^{-1/2}) J_{L_0}(q)$$

Exer: Show that

① $J_{\overbrace{\circ\circ\cdots\circ}^{n}}(q) = (q^{1/2} + q^{-1/2})^{n-1}$

② $J_{3_1}(q) = -q^4 + q^3 + q$

③ $J_{4_1}(q) = q^2 - q + 1 - q^{-1} + q^{-2}$

④ Choose your favorite

Exer: Prove:

If we swap the orientation of the link $L \mapsto \bar{L}$, then $J_{\bar{L}}(q) = J_L(q^{-1})$.

Cor: $3_1 \neq \bar{3}_1$

Outline:

1. Physical ideas & some history
2. TQFTs for low dimensions
3. Witten-Reshetikhin-Turaev inv.
4. Geom. of 3-mflds & semiclassical
5. Arithmetic aspects Habiro rings.

Historical timeline:

~'28 Alexander defn's poly. knot inv.
~'35 Reidemeister (Franz, de Rahm) defn torsion
~'61 Milnor relates Alexander poly. to Riedemeister torsion
~'70 Conway describes Alexander poly. comb.
~'71 Ray-Singer defn analytic torsion
~'77 Schwarz relates to TQFT.
~'78 Cheeger-Müller prove RS = R
~'79 Thurston studies geom. of 3-mflds
~'85 Jones defn new poly. knot inv.
~'88 Atiyah defines TQFT
~'89 Witten explains Jones poly as TQFT
~'90 Reshetikhin-Turaev realise Witten's theory.
~'93 Melvin-Morton-Rozanski give precise semiclassical conj
~'95 Kashaev states volume conj
~'96 Bar Natan - Garoufalidis prove MMR
~'99 Murakami-Murakami relate Kashaev to Jones

Witten interpretted Jones' inv. as an infinite dimensional int.

$$J_K(q) = \int_{\mathcal{A}/\mathcal{G}} e^{CS(A)/\hbar} \operatorname{tr}_{\rho_2} \operatorname{hol}_A(K)\, DA$$

where
- $\mathcal{A}$ = space of $SU(2)$-con.'s on S^3
- $\mathcal{G}$ = gauge transform.
- $CS(A) = \int_{S^3} \operatorname{tr}(dA\wedge A + \frac{2}{3} A\wedge A\wedge A)$
- $\operatorname{tr}_{\rho_2}\operatorname{hol}_A(K)$ is the trace of the holonomy along the knot.
- DA ???

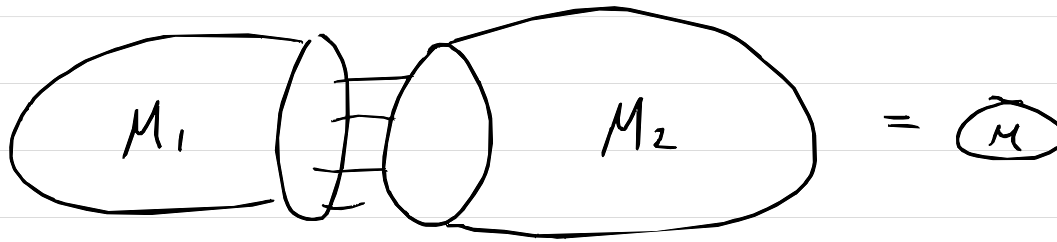
where $q = e^{\hbar}$ & $\hbar = \frac{2\pi i}{k}$ $k\in\mathbb{Z}$.

<u>Note:</u> $CS : \mathcal{A}/\mathcal{G} \to \mathbb{C}/(2\pi i)^2\mathbb{Z}$

$$\Rightarrow e^{CS(A)/\hbar} : \mathcal{A}/\mathcal{G} \to \mathbb{C}^\times.$$

<u>Q:</u> What have we gained ??

These kind of integrals should sat. cutting & gluing.



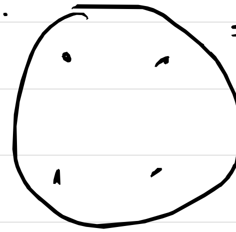
$$\int_{\mathcal{A}_M/\mathcal{G}_M} e^{S(A)/t} DA$$

$$= \int_{\mathcal{A}_\Sigma/\mathcal{G}_\Sigma} \left(\int_{\mathcal{A}_{M_1}(a)/\mathcal{G}_1} e^{S(A_1)/t} DA_1 \right) \left(\int_{\mathcal{A}_{M_2}(a)/\mathcal{G}_2} e^{S(A_2)/t} DA_2 \right) Da$$

Symplectic reduction & Duistermaat-Heckmann reduce these calculations to finite linear alg.

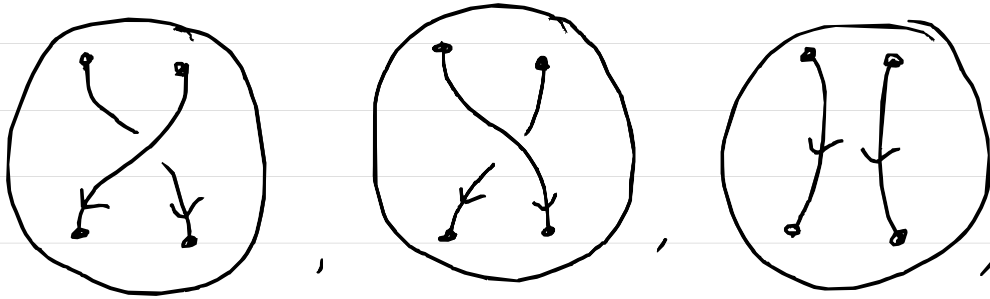
Witten made the following obs:

The vector space $Z(\Sigma_{0,4})$ of bounding cond. for $\Sigma_{0,4}$ (desc. by CFT) has dim = ?.



$= \Sigma_{0,4}$

Hence, we see that



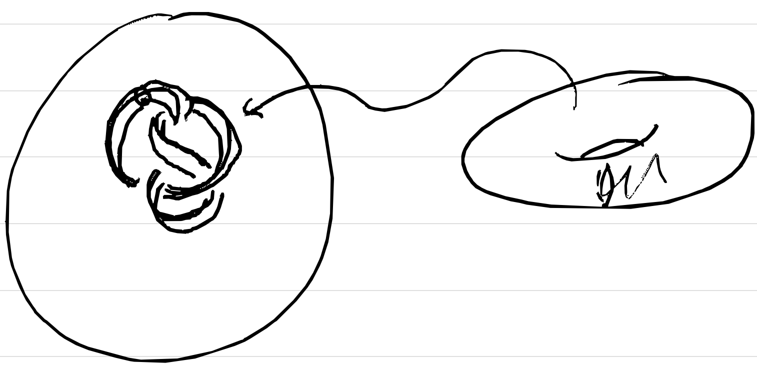
$\in Z(\Sigma_{0,4})$ must sat. a linear relation. Some additional considerations (coming from braid grp rep. of $Z(\Sigma_{0,4})$ from CFT) give Jones' skein relation.

Therefore, Witten recovers Jones' inv. but the interpretation suggests that there exists an extension of Jones' inv. by changing ρ_2, the 3-mfld S^3, $SU(2)$, & these should satisfy cutting & gluing.

$\underline{\underline{\text{Thm}}}$: (Lickorish, Wallace)

Every closed 3-mfld is obtained from surgery on a link, i.e.

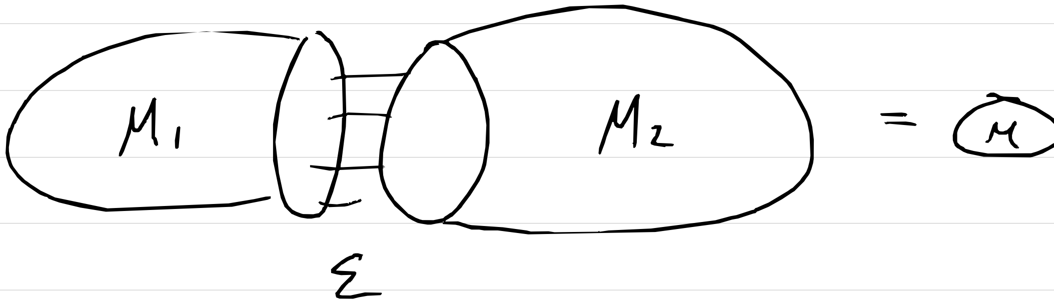
$$M = \left(S^3 \setminus L\right) \underset{\substack{\uparrow \\ \text{mapping class grp.}}}{\cup} \left(\mathbb{D} \times S^1\right)^{\# \text{comp } L}$$



Hence, Witten predicts invariants of closed 3-mflds coming from the Jones polynomial & gluing.

2 TQFTs in low dim.

Recall: Path integrals with local actions, should sat. gluing.



$$\int_{\mathcal{A}_M/\mathcal{G}_M} e^{S(A)/\hbar}\, DA$$

$$= \int_{\mathcal{A}_\Sigma/\mathcal{G}_\Sigma} \left(\int_{\mathcal{A}_{M_1}(a)/\mathcal{G}_1} e^{S(A_1)/\hbar}\, DA_1 \right) \left(\int_{\mathcal{A}_{M_2}(a)/\mathcal{G}_2} e^{S(A_2)/\hbar}\, DA_2 \right) Da$$

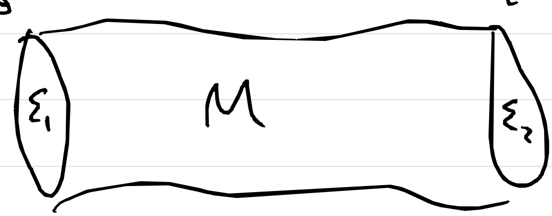
Localisation reduces to fin. dim. linear alg.

This was axiomatised by Atiyah.

Def^{n}: Let $\underline{\mathrm{Cob}}_{n+1}$ denote the category of cobordisms in $n+1$ dim.

- $\mathrm{ob}(\underline{\mathrm{Cob}}_{n+1}) = \{n\text{-dim. man.}\}$
- $\mathrm{Hom}_{\underline{\mathrm{Cob}}_{n+1}}(\Sigma_1, \Sigma_2) = \{M : \partial M = \overline{\Sigma}_1 \cup \Sigma_2\}$

diff. to $\overline{\Sigma}_1$ diff to Σ_2



This category is monoidal w.r.t $\sqcup$.

Def^{n}: An $(n+1)$-dimensional TQFT is a monoidal functor
$$Z : \underline{\mathrm{Cob}}_{n+1} \longrightarrow \underline{\mathrm{Vec}}.$$

eg - $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$

- $Z(\overline{\Sigma}) = Z(\Sigma)^*$

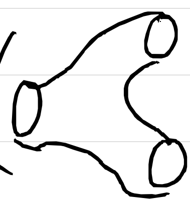
<u>Exerc</u>: Prove that $\dim(Z(\Sigma)) < \infty$.

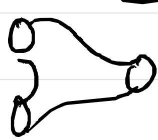
<u>Hint</u>: Use $\Sigma \times I$.

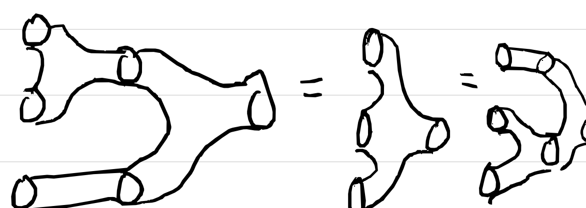
<u>Exerc</u>: Prove $(0+1)$-dim. TQFTs are classified by a fin. dim. v.s, with a non-deg. innerproduct.

<u>Exerc</u>: Prove $(1+1)$-dim. TQFTs are classified by a Frobenius alg.

<u>Hint</u>:

$Z\Big($  $\Big) \rightsquigarrow$ coproduct

$Z\Big($  $\Big) \longrightarrow$ product

Associativity $\Longleftarrow$ 

So TQFTs turn geom./top. ids into algebraic ones

(Trivial)

Example: (Dijgraff-Witten theory) 2.4

In the end we are interested in gauge theory over Lie groups like $SU(2)$ or $SL_2\mathbb{C}$. An easier starting pt. is given by taking a finite grp.

Let G be a finite grp. Then define

$$Z(\Sigma) = \mathrm{Span}_{\mathbb{C}}\{G\text{-covers of } \Sigma\}$$

$$Z(M:\Sigma \longrightarrow \emptyset)(A) = \sum_{\substack{B,\ G\text{-covers of } M \\ \text{restricting to } A \\ \text{on } \partial M = \Sigma}} \frac{1}{|\mathrm{Aut}(B)|}$$

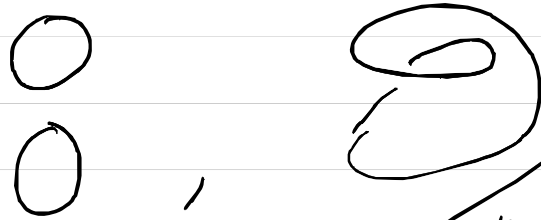
Exerc: Show that Z extends to a TQFT.

Exerc: Show that for $(1+1)$-dim that the associated Frobenius alg. is $Z(\mathbb{C}[G])$

center of the group ring.

For $G = \mathbb{Z}/2\mathbb{Z}$ (2.5)

$Z(S^1) = \mathbb{C}^2$.



Exer: Prove Mednykh's form.

$$\frac{\#\mathrm{Hom}(\pi_1(\Sigma_{g,0}), G)}{\# G} = \# G^{2g-2} \sum_{\alpha \in \mathrm{irrep}} \frac{1}{\dim \alpha^{2g-2}}$$

Ex: Compute $Z(S^3)$, $Z(S^2 \times S^1)$.

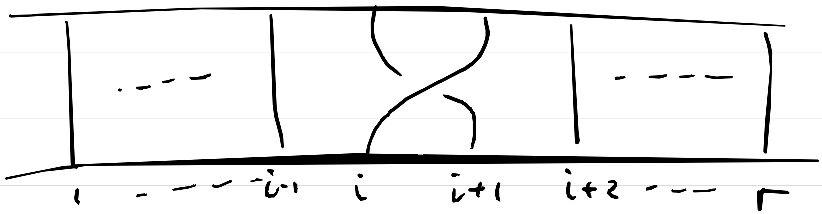
Importantly, for every $f : \Sigma \simeq \Sigma$ in the Mapping Class group. we have

$$Z(\Sigma \times I_f) : Z(\Sigma) \cong Z(\Sigma).$$

So we obtain rep. of MCG. This first appears & is interesting for $(2+1)$-dim. TQFTs.

Exer: $Z(\Sigma \times_f S^1) = \mathrm{tr}(Z(\Sigma \times I_f))$.

For (2+1)-dim., we need to understand the braid groups.

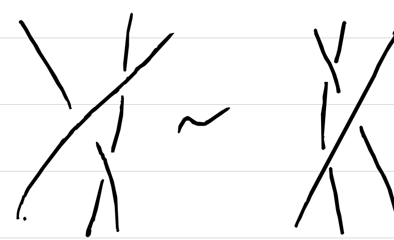


$= \sigma_i$

$$B_r = \langle \sigma_1, \dots, \sigma_{r-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$$

$$\boxed{\beta_1 \circ \beta_2 = \frac{\beta_2}{\beta_1}}$$

<u>Exerc:</u> Show that $\sigma_i \sigma_i^{-1} = 1$ & $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ corresp. to Riedermeister II & III.



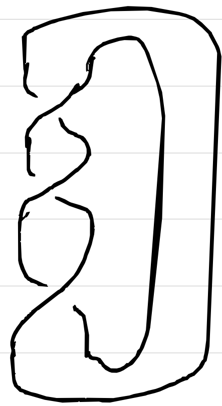
$B_r = MCG(\Sigma_{0,r})$

<u>Thm:</u> (Alexander + Markov) Every link L can be expressed as the closure of a braid, & two links are equiv. by seq. ① $\beta \sim \alpha \beta \alpha^{-1}$ ② $\beta \sim c_n(\beta) \sigma_n^{\pm 1}$

$c_n : B_n \to B_{n+1}$

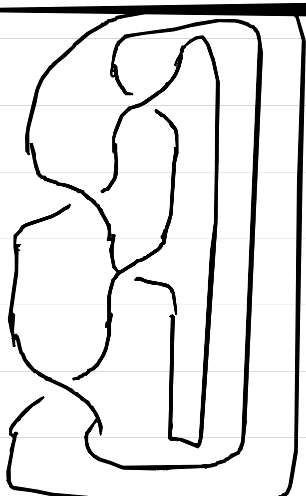
Ex:

3, 2

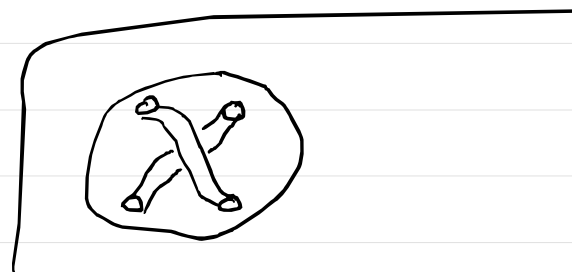


σ_1^3 in B_2

4, 2



$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$ in B_3



We want to construct braid reps. to const. inv. of 3-mflds. We will use quantum grps. to do this.

Rmk: Braids are already appear as some kind of extended TQFT.

3 WRT - inv.

Witten conjectured & outlined the construction of a (2+1)-dim TQFT associated to a Lie group. Reshetikhin-Turaev constructed this using quantum groups.

sl_2

Ex: $U_q sl_2$ $\left[H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right]$

"$K = \exp(H)$"

$$KX = q^{\frac{1}{2}} X K, \quad KY = q^{-\frac{1}{2}} Y K,$$

$$XY - YX = \frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}}$$

If $q^r = 1$, then add $X^r = Y^r = 0$, $K^{4r} = 1$,

This is a Hopf alg. $\Delta(X) = X \otimes K + K^{-1} \otimes X$

$S(X) = -q^{\frac{1}{2}} X$ $\quad \Delta(Y) = Y \otimes K + K^{-1} \otimes Y$

$S(Y) = -q^{-\frac{1}{2}} Y$ $\quad \Delta(K) = K \otimes K$

$S(K) = K^{-1}$ $\quad \varepsilon(X) = \varepsilon(Y) = 0, \quad \varepsilon(K) = 1.$

Reps $V^{(k)} = \mathrm{Span}\{e_i \mid i = -m, \dots, m\}$ $\quad m = \frac{k-1}{2}$

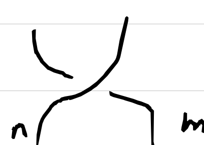
$$Xe_j = \frac{q^{\frac{m+j+1}{2}} - q^{-\frac{m+j+1}{2}}}{q^{1/2} - q^{-1/2}}, \quad Ye_j = \frac{q^{\frac{m-j+1}{2}} - q^{-\frac{m-j+1}{2}}}{q^{1/2} - q^{-1/2}} e_{j-1}, \quad Ke_j = q^{j/2} e_j$$

Quantum groups come with R-matrices which give rise to a braiding, in particular, reps. of the braid grp. For Sl_2, we obtain

$$R = \frac{1}{4r} \sum_{n,a,b} (-1)^n \frac{q^{n(n-3)/4 - ab - bn + an - 1} (1-q)^{2n}}{(q;q)_n} Y^n K^b \otimes X^n K^a$$

Exerc: Prove $(R\otimes \mathrm{Id})(\mathrm{Id}\otimes R)(R\otimes \mathrm{Id}) = (\mathrm{Id}\otimes R)(R\otimes \mathrm{Id})(\mathrm{Id}\otimes R)$.

So reps. of quantum groups form a braided monoidal category. We can the def inv. of links using these reps. by taking a quantum trace when we colour a link by reps.



Then

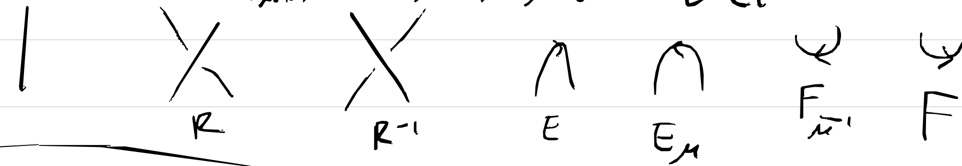
$$R(e_k \otimes e_l) = \sum_{i=-m}^{m} \sum_{j=-n}^{n} \sum_{p=0}^{\min(m-i,\, j+n)} \delta_{l,i+p}\, \delta_{k+p,j}$$

$$(-1)^p q^{ij - \frac{p}{2}(m+n) - (i-j)p - p(p+1)/2} \frac{(q;q)_{m+l}(q;q)_{n-k}}{(q;q)_{m+i}(q;q)_p (q;q)_{n-j}} e_i \otimes e_j$$

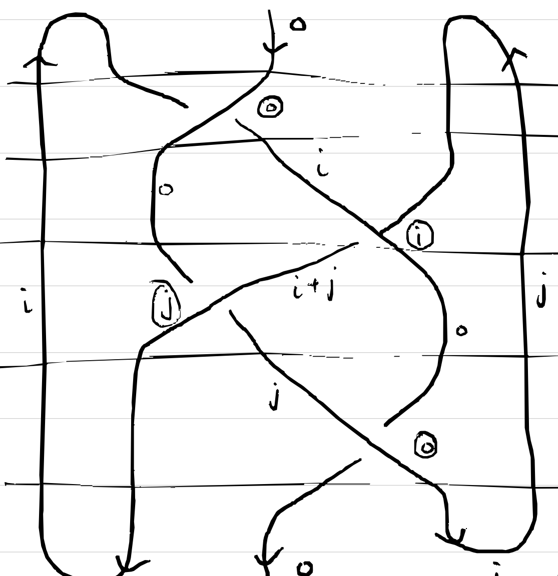
Quantum groups have alg. struct. underlying $\mathrm{Cob}(2+1)$

$E_\mu(f\otimes x)=f(\mu(x))$

$F_\mu(1)=\sum e^i\otimes\mu(e_i)$ $\quad \mu(e_i)=q^i e_i$



Ex: 4_1, $\quad 2n+1=N$



$E\otimes \mathrm{Id}\otimes E_\mu$

$\mathrm{Id}\otimes R\otimes \mathrm{Id}\otimes \mathrm{id}$

$\mathrm{Id}\otimes \mathrm{id}\otimes R\otimes \mathrm{id}$

$\mathrm{Id}\otimes R\otimes \mathrm{Id}\otimes \mathrm{id}$

$\mathrm{Id}\otimes \mathrm{id}\otimes R\otimes \mathrm{id}$

$F_{\mu^{-1}}\otimes \mathrm{Id}\otimes F$

$$= \sum_{i,j} \mu^j_j\, R^{i0}_{0i}\,(R^{-1})^{ij}_{i+j\,0}\,R^{0\,i+j}_{i\,j}\,(R^{-1})^{0j}_{i0}\,(\mu^{-1})^i_i$$

$$J_{4_1,N}(q) = \sum_{i,j}^{2n}(-1)^j q^{-(2n+1)i+j(j+1)/2}\frac{(q;q)_{i+j}(q;q)_{2n}}{(q;q)_i(q;q)_j(q;q)_{2n-i-j}}$$

$$= \frac{q^{-(N-1)/2}}{1-q}\sum_{k=0}^{N-1} q^{-Nk}\frac{(q;q)_{N+k}}{(q;q)_{N-1-k}}$$

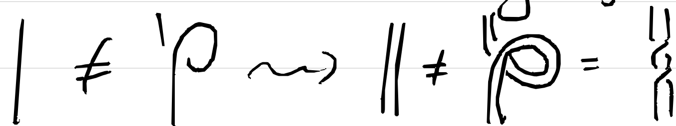
$$= \frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}}\sum_{k=0}^{N-1} q^{-kN}(q^{N-1};q^{-1})_k(q^{N+1};q)_k$$

Rmk: Normalizin

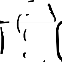
Exerc: Compute for 3, & check it agrees with $\frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}}\sum_{k=0}^{N-1}(q^{1-N};q)_k(q^{N+1};q)_k$

Recall: Lickorish, Wallace say every 3-mfld is surgery on some link.

Can describe this by framed links.

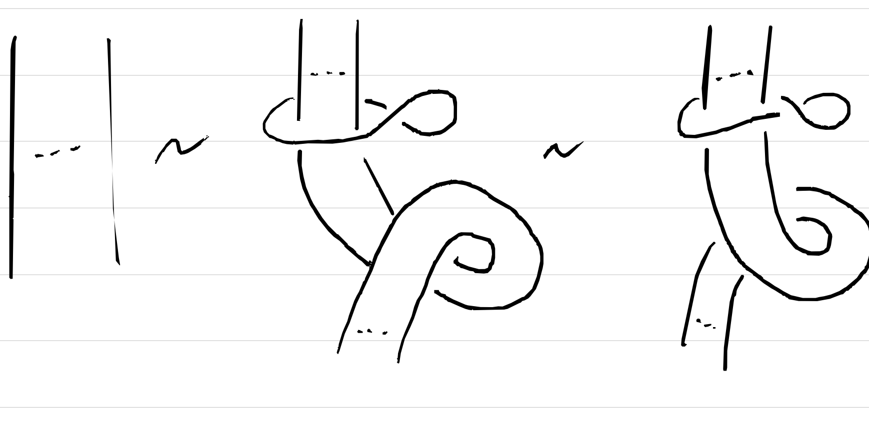


Def: linking matrix

Exerc: Show  $= q^{(N^2-1)/2} J_K$

Then Kirby describes when two surgeries are equiv. using 4d techniques.

Thm: (Kirby) Two surgeries of links are related by a sequence of moves of the form.



Therefore, to find an inv of closed 3-mflds, we just need to check this equiv.

Witten already suggested the answer & Reshetikhin-Turaev defined & proved it.

Thm (RT) If M is surgery on a link L, then. σ_L = signature of linking matrix.

$$Z_M(\zeta) = e\left(\tfrac{-3}{8}\sigma_L\right) \left(\frac{2e(-1/8)(q^{1/2}-q^{-1/2})}{\Sigma\, q^{k^2/4}}\right)^{\ell} \sum_{n_i=0}^{m-1} J_{L,n}(\zeta) \prod_{j=1}^{\ell} \frac{q^{n_j/2}-q^{-n_j/2}}{q^{1/2}-q^{-1/2}}$$

$q^{\frac{3\sigma_L}{4}}$

is a topological inv. of M.
i.e. is inv. under Kirby moves.

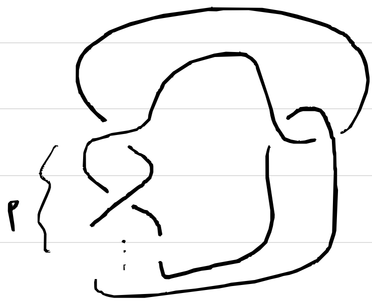
To give examples, let me state 2 theorems of Habiro

Thm: $\forall$ knots K $\exists$ poly. $C_{K,k}(q) \in \mathbb{Z}[q^{\pm 1}]$ s.t.

$$J_{K,n}(q) = \Sigma\, C_{K,k}(q)\,(q^{n+1};q)_k\,(q^{1-n};q)_k$$

Ex: (Masbaum) $C_k(K_p; q)$

$$= q^k \sum_{0 \le s_1 \le \cdots \le s_p = k} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \binom{s_{i+1}}{s_i}_q$$



$= K_p$

$K_1 = 3_1$

$K_{-1} = 4_1$

Ex: (Beliakova–Lê) If K is a knot then if $M = K(-1, b)$

$$(1-q) Z_M(q) = \sum_{k=0}^{\infty} C_k(K; q)(q^{k+1}; q)_{k+1} \cdot \sum_{0 \le k_1 \le \cdots \le k_b = k} \prod_{j=1}^{b-1} q^{k_j(k_j+1)} \binom{k_{j+1}}{k_j}_q$$

These can effectively computed since they are q-hyp. geom, eg if

$a_{n+1} = R(q^n, q) a_n$, & Σa_n then

sum = temp = a_0; for($k = 1$, .., , temp = temp $R(q^k, q)$;
sum = sum + temp); sum

Exerc: ① Write a program to compute $J_{4,n}(q)$.

Try to beat

② Compute the values $J_{4,1,n}(e^{2\pi i/n})$

for $n = 100, \ldots, 110$.

③ Try to determine the asymp. behavior of these values numerically.

Exerc: Using the formula of Bel---Lê do the same for a closed manifold. ③ is very hard in comparison.

4 geom. of 3-mflds & semiclassical.

<u>Recall:</u> Witten's integral

$$Z(\hbar) = \int_{\mathcal{A}/\mathcal{G}} e^{CS(A)/\hbar}\, DA$$

Laplace & Riemann (+ Feynmann) showed us how to compute asymp. of integrals of the form

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<u>Main idea:</u> deform contour to have $\mathrm{Re}(CS(A)/\hbar)$ reach a unique maximum. Needs take a critical pt. (Method of steepest descent.)

$$CS'(A) = 0 \iff A \text{ is a flat con.}$$

$$\frac{\{\text{flat } SL_2\mathbb{C} \text{ con.}\}}{\text{gauge}} \xleftrightarrow{\text{bij.}} \mathrm{Hom}(\pi_1(M), SL_2\mathbb{C})/SL_2\mathbb{C}$$

" $R_{M, SL_2\mathbb{C}}$

Hence, we expect

$$Z(\hbar) \sim \sum_{\theta = CS(A_\rho)\text{ flat. con.}} e^{\theta/\hbar}\, \Phi_\theta(\hbar).$$

What makes $SL_2\mathbb{C}$ a particularly fun grp. for 3-mflds is the fact that $\mathrm{Isom}(\mathbb{H}^3) \cong PSL_2\mathbb{C}$.

Conj: (Kashaev, Murakami^2)

If K is a hyp. knot, then

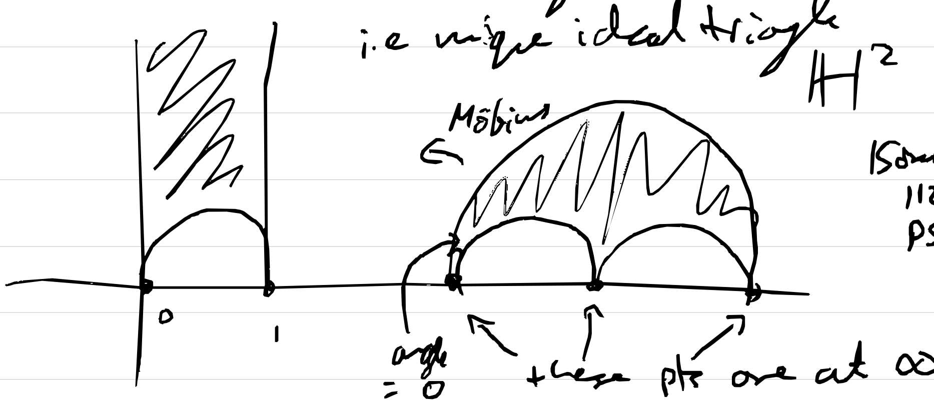
$$J_{K,n}(e^{2\pi i/n}) \underset{n\to\infty}{\sim} \exp\left(\frac{\mathrm{Vol}_{\mathbb{C}}(S^3 \setminus K)}{2\pi i} n\right) n^{3/2}\, \mathbb{C}[\![\tfrac{1}{n}]\!].$$

~~How to compute??~~

Thurston gave a combinatorial & computational approach to studying 3-mflds via ideal triang.

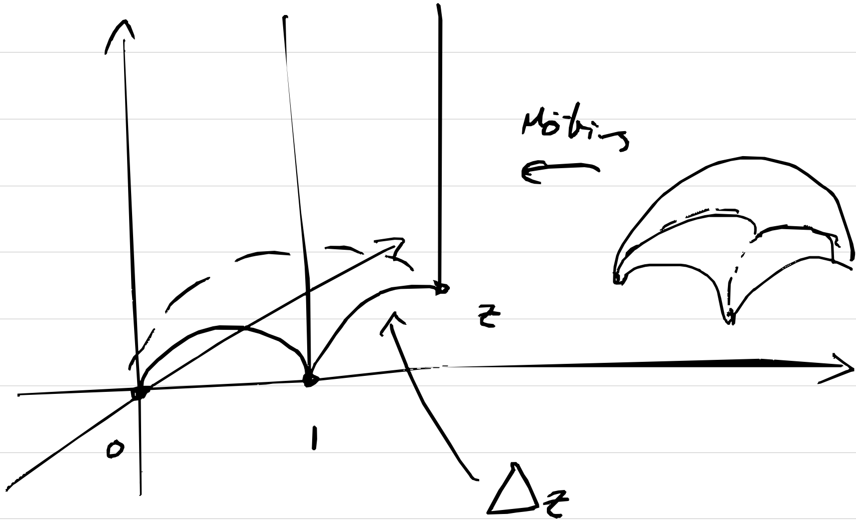
Recall: Ideal triangle

i.e unique ideal triangle $\mathbb{H}^2$



$\mathrm{Isom}(\mathbb{H}^2) \cong PSL_2\mathbb{R}$.

$\mathbb{H}^3$



Therefore, have $\dim_{\mathbb{C}} = 1$ moduli of ideal tetrahedra

$$z' = \frac{1}{1-z}$$

$$\Delta_z \cong \Delta_{z'} \cong \Delta_{z''} \qquad z'' = 1 - z^{-1}$$

Exerc: Show that

$$\mathrm{Vol}(\Delta_z) = \Lambda(\arg(z)) + \Lambda(\arg(z')) + \Lambda(\arg(z''))$$

$$= \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-z)\log|z|$$

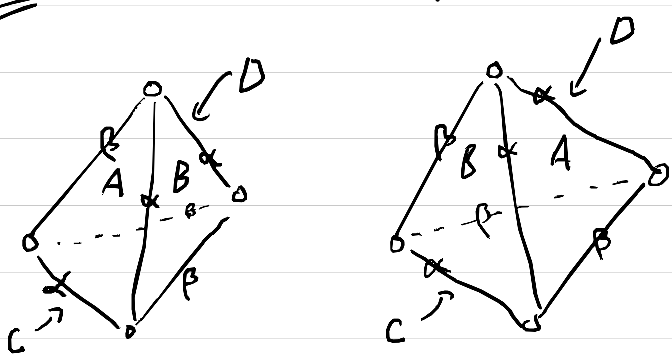
where $\Lambda(x) = -\int_0^x \log|2\sin(t)|\,dt$ is Lobachefsky fn. $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ $" = -\int_0^z \log(1-w)\frac{dw}{w}"$

Thm (Thurston) If M is a complete hyp. manifold, then ∂M is Euclidean.

Cor: If M is a 3-mfld $\partial M = \sqcup S^1 \times S^1$.

We can ideally triangulate these 3-mflds.

Ex: $M = S^3 \setminus 4_1$



Exerc: Download SnapPy & play eg. compute vol. of some knots.

In fact, from a link diagram (with at least one over & under crossing for each component) there is an ~~almost~~ canonical triangulation.

Recall:
$$R(e_k \otimes e_\ell) = \sum_{i=-m}^{m} \sum_{j=-n}^{n} \sum_{p=0}^{\min(m-i,\, j+n)} \delta_{\ell, i+p}\, \delta_{k+p, j}\, (-1)^p q^{ij - \frac{p}{2}(m+n) - (i-j)p - p(p+1)/2} \frac{(q;q)_{m+\ell}(q;q)_{n-k}}{(q;q)_{m+i}(q;q)_p (q;q)_{n-j}} e_i \otimes e_j$$

Exerc: ① $$(x;q)_\infty = \exp\left(\sum_{k=0}^{\infty} \frac{x^k}{(q^k-1)k}\right)$$

② $$(x;e^t)_\infty \overset{t\to 0}{\sim} \exp\left(\sum_{\ell=0}^{\infty} \frac{B_\ell}{\ell!} \mathrm{Li}_{2-\ell}(x) t^{\ell-1}\right)$$

③ $$R(e_i \otimes e_j) \overset{t\to 0}{\sim} \qquad x = \frac{i}{t} \quad y = \frac{j}{t} \quad z = \frac{p}{t} \qquad {}^{\shortparallel}\ \Phi(x;t)$$

$= \int dz \exp(W/t) \ldots$

$$\int dz \exp\Big[\Big(\pi i \log z + \log x \log y - (\log x - \log y)\log z - \log z^2/2 - \frac{\pi^2}{6} + \mathrm{Li}_2(t_m x) + \mathrm{Li}_2(z) + \mathrm{Li}_2(t_n y^{-1}) - \mathrm{Li}_2(t_m x z) - \mathrm{Li}_2(t_n y^{-1} z)\Big)\Big/ t\Big] \ldots$$

Therefore, to each crossing we have a potential W. There are 5 dilogarithms. These come from 5 tetrahedra coming from an octahedron.

Thm (Yoon) If we have a link diagram s.t all comp. have an over & under crossing, then the critical points of $\sum_{\text{crossing}} W$ is a 2-dim fibration over $R_{S^3 \setminus L, SL_2\mathbb{C}}$

Hence, naively applying Euler-Maclaurin or Poisson summation to the state sum + stationary phase gives the vol. conj.

<u>Ex</u>: 4_1

$$J_0(q) = \sum_{n=0}^{\infty} (-1)^n q^{-n(n+1)/2} (q;q)_n^2$$

Then $z\frac{d}{dz}\left(-2\mathrm{Li}_2(z) - \frac{\log(z)^2}{2} + \frac{\log z}{2}\,2\pi i\right) = +2\log(1-z) - \log z + \pi i$

$$\Rightarrow (1-z)^2 = -z \quad \Rightarrow \quad z = e(\pm 1/6)$$

Then value $= \frac{\pi^2}{18} \pm 2.0299\ldots i$

"$\mathrm{vol}(S^3 \setminus 4_1)$".

In fact, one expects much more. Using the "resummation" of $\Phi(x;\hbar)$, one finds that the resumm. of the asymp. of the volume conj is given by fin. dim. integrals of $\Phi(x;\hbar)$.

Let $\Omega(x;\hbar)$ denote the matrix of this.

(Quantum modularity) 4.7

Conj: (Garoufalidis - Zagier)
Gu - Mariño

$$\begin{pmatrix} J(\tilde{x};\tilde{q}) \\ \vdots \\ J(\tilde{q}^n\tilde{x};\tilde{q}) \end{pmatrix} = \Omega(z;\tau) \begin{pmatrix} J(x;q) \\ \vdots \\ J(q^n x;q) \end{pmatrix}$$

where $q = e(\tau)$, $\tilde{q} = e(-1/\tau)$
$x = e(z)$, $\tilde{x} = e(z/\tau)$

Th (Fantini-W.) True for 4_1 & 5_2.

Q: Therefore, why isn't the volume conj. proved?

A: One needs good control over the potential function & the steepest descent contours. Non-compact spaces mean ∞ becomes important.

⑤ Arithmetic aspects

Last time: We saw that semi-classical /asymptotic behavior gave asymp. series associated to flat $SL_2\mathbb{C}$ conn on 3-mnflds.

Therefore, one expects to upgrade TQFT to give a class

$$[\omega] \in \mathcal{H}^*(R_{M,G}, \hbar)$$

moduli space of flat con. $\cong \mathrm{Hom}(\pi_1(M), G)/G$

Q: What is this cohomology theory?

A: Work in progress... but is definitely called Habiro cohomology.

5.2

Recall: Habiro showed that $\forall$ knots K, $\exists$ $C_{K,k}$ s.t

$$J_{K,n}(q) = \sum_{k=0}^{\infty} C_{K,k}(q)\,(q^{n+1};q)_k\,(q^{1-n};q)_k.$$

Hence, for volume conj. we have $q^n = 1$, we can consider

$$J_{K,0}(q) = \sum_{k=0}^{\infty} C_{K,k}(q)\,(q;q)_k^2. \leftarrow \text{Kashaev invar.}$$

Def: (Habiro ring)

$$\mathcal{H}_{\mathbb{Z}} := \varprojlim_n \mathbb{Z}[q]/(q;q)_n\mathbb{Z}[q]$$

Ex: All elements of $\mathcal{H}_{\mathbb{Z}}$ can be written (non-canonically) as

$$\sum a_k(q)\,(q;q)_k \qquad a_k \in \mathbb{Z}[q]$$

Cor: Kashaev inv. are in $\mathcal{H}_{\mathbb{Z}}$.

Exerc: Show that $q^{-1} \in \mathcal{H}_{\mathbb{Z}}$.

Question: What are the units $\mathcal{H}_{\mathbb{Z}}^{\times}$?

Notice that, for all roots of unity ζ, $(\zeta;\zeta)_k = 0$ for $k >> 0$.
Therefore we can map $ev_\zeta : \mathcal{H}_\mathbb{Z} \to \mathbb{Z}[\zeta]$.

<u>Recall</u>: The p-adic numbers

$$\mathbb{Z}_p = \varprojlim_m \mathbb{Z}/p^m\mathbb{Z} \quad , \quad \mathbb{Q}_p = \mathbb{Z}_p[\tfrac{1}{p}].$$

$$\mathbb{Z}_p = \mathbb{Z}[\![p]\!] \quad \& \quad \mathbb{Q}_p = \mathbb{Z}(\!(p)\!).$$

<u>Ex</u>: $\frac{-1}{6} = \frac{1}{1-7} = 1 + 7 + 7^2 + \cdots \in \mathbb{Z}_7$.

We have a norm $|x|_p = p^{-v_p(x)}$, i.e. stores the first power of p, in $\mathbb{Q}_p$.
This can be extended to other field extensions of $\mathbb{Q}_p$.

<u>Exerc</u>: Let ζ_{p^d} be a prim. p^d-th root of unity, i.e. a root of
$$x^{p^{d-1}(p-1)} + x^{p^{d-1}(p-2)} + \cdots + x^{p^{d-1}} + 1.$$
Show that $|\zeta_{p^d} - 1|_p < 1$. <u>Hint</u>: $|x^d|_p = |x|_p^d$ if $|px|_p = |x|_p\, p'$

Thm (Habiro): For $\zeta_{mm'} = \zeta_m \zeta_{m'}$ $(m,m')=1$
$\zeta_{p^{d+1}}^{p} = \zeta_{p^d}$

$$\mathcal{H}_{\mathbb{Z}} \underset{ev}{\cong} \{ (f_m) \in \prod_m \mathbb{Z}[\zeta_m][[q-\zeta_m]] \mid$$

$$f_m(q-\zeta_{pm}+\zeta_{pm}-\zeta_m) = f_{mp}(q-\zeta_{pm}) \in \mathbb{Z}_p[\zeta_{pm}][[q-\zeta_{pm}]] \}.$$

In algebraic geometry, cohomology of varieties/$\mathbb{Q}$ carry a lot of structure. Hodge decomp., quasi-unipotent monodromy, & important for us is Frobenius morphisms.

$$\varphi_p : H^n(X;\mathbb{Q}_p) \longrightarrow H^n(X;\mathbb{Q}_p)$$

that generalise the map $\varphi_p(x) = x^p$.

see Scholze & Wagner

"**Defn**" $\mathcal{H}_X^n \cong \{ (f_m) \in \prod_m H^n(X;\mathbb{Q})[\zeta_m][[q-\zeta_m]] \mid$

$$f_m(q-\zeta_{pm}+\zeta_{pm}-\zeta_m) = \varphi_p f_{mp}(q-\zeta_{pm}) \in H^n(X;\mathbb{Q}_p)[\zeta_{pm}][[q-\zeta_{pm}]] \}$$

5.5

Ex: $\sum_{k=0}^{\infty} \binom{2k}{k}_q t^k = F(t,q)$

We define $f_m = F(t^{1/m}, \zeta_m + (q - \zeta_m))$

Notice that $f_1 = \sum_{k=0}^{\infty} \binom{2k}{k} t^k + O(q-1)$

$$= \frac{1}{\sqrt{1-4t}} + O(q-1)$$

Exerc: ① Compute $\varphi_p\left(\frac{1}{\sqrt{1-4t}}\right) + O(p^3)$ $\in \mathbb{Z}[t, \frac{1}{\sqrt{1-4t}}]_p^\wedge$ where $\varphi_p(t) = t^p$. $\boxed{p \neq 2.}$

② Compute $f_1 + O(q-1)^5$

③ check that after sub. reexp. to $q = \zeta_5$ agrees

Ex: $\sum_{k=0}^{\infty} \binom{2k}{k}_q^2 t^k = F(t,q)$

want to define again $f_m = F(t^{1/m}, \zeta_m + (q - \zeta_m))$

Note that $\sum_{k=0}^{\infty} \binom{2k}{k}^2 t^k$ ← elliptic fn.

$$= \int \frac{1}{\sqrt{(1-z)(1-z^{-1}t)}} \frac{dz}{z}$$

Therefore replace $\sum_{k=0}^{\infty} \binom{2k}{k}^2 k^m t^k \rightsquigarrow \left(t\frac{d}{dt}\right)^m \left[\int \frac{1}{\sqrt{(1-z)(1-z^{-1}t)}} \frac{dz}{z}\right]$

to define f_m via $F(t^{1/m}, \zeta_m + (q - \zeta_m))$.

5.6

Exerc: Compute φ_p for H^1 of this elliptic curve to some acc.

Finally, where does the line bundle $\mathcal{L}$ come in?

This stores the Chern-Simons fn. eg the volume term.

$$\exp(CS(A)/\hbar).$$

Thm (Kontsevich-Soibelman) $Q = Q^T \in M_{N\times N}(\mathbb{Z})$.

$$\sum_k (-q)^{\frac{kQk}{2}} \frac{1}{(q;q)_{k_1} \cdots (q;q)_{k_N}} t_1^{k_1} \cdots t_N^{k_N}$$

$$= \prod_i \prod_{j=1}^{\infty} (q^j t^i ; q)_\infty^{c_{i,j}}$$

for fixed $i = i_1 \dots i_N$ there are finite j s.t. $c_{i,j} \neq 0$.

Exerc: try some $Q = Q^T$ & compute c_{ij} for some i.

Thm: If $X_0 \subseteq X$ is an isolated pt. eg geom. (i.e. defines a number field) then

Garoufalidis-Scholze-Wheeler-Zagier

$$\Phi_{geom}(\hbar) \in \mathcal{L} \in \mathrm{Pic}(\mathcal{H}_{X_0}^{\circ}).$$

Summary:

- Jones discovered an invariant of links from braid reps.

- Witten interpreted this as an infinite dim. integral

- Reshetikhin-Turaev showed that Witten's theory would be mathem. const.

- Conj. on the asymp of WRT inv. that one expected from the ∞-dim integral.

- Habiro shows arithmetic prop. of these invariants.

Current / Future: Expect even more from Witten's integral. Expect a whole cohomology theory + classes.