

# Exercises for lecture 1

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*Exercise 1.* Given a supervector space  $V$ , define the *fermion parity operator* as  $(-1)_V^F(v) = (-1)^{|v|}v$ , where  $|v| \in \{0, 1\}$  is the degree of  $v$ . Let  $Z: \text{Bord}_{d,d-1}^X \rightarrow \text{sVect}_{\mathbb{C}}$  be a TQFT. Show that  $Z$  is bosonic if and only if  $(-1)_{Z(Y)}^F = \text{id}_{Z(Y)}$  for all objects  $Y^{d-1}$  of  $\text{Bord}_{d,d-1}^X$ .

*Exercise 2.* Recall that  $\text{Bord}_{n,n-1}^{X(n)}$  is the bordism category associated to  $X(n) \rightarrow BO(n)$ . Let  $f: Y_1^{n-1} \rightarrow Y_2^{n-1}$  be an  $X(n-1)$  structure preserving diffeomorphism between closed manifolds.

- Define an associated morphism  $Y_1 \rightarrow Y_2$  in the bordism category. (Hint: as a manifold with boundary you can take  $Y_2 \times [0, 1]$ )
- Show that this is an isomorphism in the bordism category.

*Exercise 3.* Show that the braiding  $\beta_{V,W}: V \hat{\otimes} W \rightarrow W \hat{\otimes} V$  given by  $v \otimes w \mapsto (-1)^{|v||w|}w \otimes v$  makes  $\text{sVect}_k$  into a symmetric monoidal category.

*Exercise 4.* In the lecture we defined  $BSO \rightarrow BO$  as the universal cover and  $BSO(n) \rightarrow BSO$  as the homotopy pullback along  $BO(n) \rightarrow BO$ . Show that  $BSO(0)$  is a discrete space with two points.

*Exercise 5.* For  $n > 0$  there exists a topological group  $SO(n)$  of which the classifying space is  $BSO(n)$ . There is also a group homomorphism  $SO(n) \rightarrow O(n)$  that on classifying spaces gives  $BSO(n) \rightarrow BO(n)$ .

The construction for  $n > 0$  of a topological group  $\text{Spin}(n)$  and a map  $\text{Spin}(n) \rightarrow O(n)$  of which the classifying space is  $B\text{Spin}(n) \rightarrow BO(n)$  is as follows. Let  $Cl_n$  be the Clifford algebra generated by vectors  $v \in \mathbb{R}^n$  satisfying the relation

$$\frac{vw + wv}{2} = \langle v, w \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the standard inner product. The *pin group* is the subgroup of  $Cl_n^\times$  generated by vectors  $v \in \mathbb{R}^n$  of length one. The *spin group*  $\text{Spin}(n)$  is the subgroup of the pin group in which we only allow even vectors. There is a homomorphism  $p: \text{Spin}(n) \rightarrow O(n)$  which maps  $x \in \text{Spin}(n) \subseteq Cl_n$  to the orthogonal linear map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  given by  $v \mapsto vxv^{-1} \in \mathbb{R}^n \subseteq Cl_n$ .<sup>1</sup>

We now consider  $n = 2$

- Show that  $\text{Spin}(2) = \{\cos \theta + \sin \theta e_1 e_2 : \theta \in \mathbb{R}\} \subset Cl_2^\times$ , where  $e_1, e_2$  is the standard basis.
- Compute that  $(e_1 e_2)^2 = -1$  and conclude that  $\text{Spin}(2) \cong SO(2)$ .
- Show that  $p(\cos \theta + \sin \theta e_1 e_2)$  is given by a rotation in the plane with  $2\theta$ .
- To what subgroup of  $\pi_1(SO(2))$  does the covering space  $\text{Spin}(2) \rightarrow SO(2)$  correspond? Conclude that  $B\text{Spin}(2)$  is not  $\pi_{\geq 3}BSO(2)$ .

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<sup>1</sup>It is not immediately clear that this element of  $Cl_n$  lands in  $\mathbb{R}^n$  but it does.