

Exercises for lecture 2

Luuk Stehouwer

May 2026

Exercise 1. Let $+$ and $-$ denote the connected zero-dimensional manifold with either of its maps to the two points of $BSO(0)$. Draw pictures that show that $+$ and $-$ are dual in $\text{Bord}_{1,0}^{SO}$.

Exercise 2. Let $Z: \text{Bord}_{1,0}^{SO} \rightarrow \text{Vect}_{\mathbb{C}}$ be a one-dimensional bosonic field theory such that $Z(+)=V$. Use the previous exercise to compute

- a) $Z(+ \sqcup -)$;
- b) $Z(S^1)$.

Exercise 3. Show that a nondegenerate Hermitian pairing on an object $V = V_0 \oplus V_1 \in \text{sVect}$ is a nondegenerate sesquilinear pairing $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$ such that $V_0 \perp V_1$ and

$$\langle v, w \rangle = (-1)^{|v||w|} \overline{\langle w, v \rangle}$$

for all homogeneous $v, w \in V$. Conclude that $\langle v, v \rangle$ is real when $|v| = 0$ and imaginary when $|v| = 1$.

Exercise 4. Let \mathcal{C} be a symmetric monoidal category in which we equipped all objects with duals. Prove that if $f: y \rightarrow x$ is a morphism a formula for $f^\vee: x^\vee \rightarrow y^\vee$ is given by

$$x^\vee \xrightarrow{\sim} x^\vee \otimes 1 \xrightarrow{\text{coev}_y} x^\vee \otimes y \otimes y^\vee \xrightarrow{\text{id} \otimes f \otimes \text{id}} x^\vee \otimes x \otimes y^\vee \xrightarrow{\text{ev}_x \otimes \text{id}} 1 \otimes y^\vee \xrightarrow{\sim} y^\vee.$$

Exercise 5. Let $X(n) = \text{pt} \rightarrow BO(n)$ so that an $X(n)$ -structure is a framing, i.e. a trivialization of the tangent bundle. Show that $\text{Bord}_{2,1}^{X(2)}$ cannot be made into a dagger category. (Hint: closed genus $g > 1$ surfaces don't have framings, but surfaces with nontrivial boundary do)

Exercise 6. Show that an object of (Vect_k, \otimes) that is dualizable is necessarily finite-dimensional. Use this to solve Campbell's exercise saying that $\dim Z(Y^{d-1}) < \infty$ for all $Y \in \text{Bord}_{d,d-1}^X$.

Exercise 7. Let Isomet be the category of Hilbert spaces and isometries. Prove that a d -dimensional TQFT valued in Isomet assigns one-dimensional vector spaces to every $Y^{d-1} \in \text{Bord}_{d,d-1}$. (Hint: what are duals in Isomet)