Exercises for lecture 3

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Exercise 1. a) Prove that the determinant gives a homomorphism of Lie groups $O(n) \to O(1)$ which commutes with the block diagonal inclusion $O(n) \to O(n+1)$.

b) Show that $BO(1) \to BO$ is a section of the induced map $\det\colon BO \to BO(1)$. Conclude that $BO \cong BSO \times BO(1)$ as spaces.

c) Why is the induced map $BO(1) \to BO$ not a group homomorphism, i.e. not $\mathbb{E}_1$? (Hint: the group structure on $BO(1)$ is $\otimes$)

Exercise 2. Recall that if G is an ∞-group, a G-action on a category $\mathcal{C}$ is a functor $BG \to \mathrm{Cat}_1$, where Cat_1 is the $(2,1)$-category of categories. Show that for $G = B\mathbb{Z}/2$, a G-action is the same as an natural automorphism ϕ of $\mathrm{id}_{\mathcal{C}}$ such that $\phi_x^2 = \mathrm{id}_x$ for all $x \in \mathcal{C}$.

Exercise 3. A super Hilbert space is defined to be a Hermitian pairing $h\colon \overline{V} \otimes V \to \mathbb{C}$ in super Hilbert spaces such that $h(v,v) \in \mathbb{R}_{\geq 0}$ if $v \in V_0$ and $h(v,v) \in i\mathbb{R}_{\geq 0}$ if $v \in V_1$. Show that if h is a super Hilbert space, then

$$\overline{\overline{V}} \otimes \overline{V} \cong \overline{\overline{V} \otimes V} \xrightarrow{\overline{h}} \overline{\mathbb{C}} \cong \mathbb{C}$$

makes $\overline{V}$ into a super Hilbert space if and only if h is purely even.

Exercise 4. Recall that $\pi_{\leq 2} BO = B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$ with group structure $(w_1, w_2) \oplus (w_1', w_2') = (w_1 + w_1', w_2 + w_2' + w_1 w_1')$.

a) Check that $(w_1, 0) + (w_1, 0) = (0, w_1^2)$ and $\ominus(w_1, 0) = (w_1, w_1^2)$.

b) Let $\mathcal{C}$ be a category with O-action $(\overline{(.)}, (-1)^F)$. Show that in the $\mathbb{Z}/2$-action given by the diagonal restriction

$$B\mathbb{Z}/2 \xrightarrow{\Delta} B\mathbb{Z}/2 \times B\mathbb{Z}/2 \to BO \times BO \xrightarrow{\oplus} BO$$

the generator $g \in \mathbb{Z}/2$ acts by the identity but the identification $\mathrm{id} = g^2 \cong \mathrm{id}$ is changed by $(-1)^F$.

Exercise 5. Consider the O-action on $\mathrm{sVect}_{\mathbb{C}}$ given by the $\mathbb{Z}/2$-action $V \mapsto \overline{V}$ and the $B\mathbb{Z}/2$-action $(-1)^F$.

a) Prove that $\mathbb{Z}/2$-fixed points for the restricted $O(1)$-action $O(1) \hookrightarrow O$ on $\mathrm{sVect}_{\mathbb{R}}$ are given by a grading-preserving real structure on a complex vector space.

b) Prove that $\mathbb{Z}/2$-fixed points for the $O(1)$-action $O(1) \hookrightarrow O \xrightarrow{\ominus} O$ on $\mathrm{sVect}_{\mathbb{R}}$ are given by a real structure on the even part and a quaternionic structure on the odd part. Conclude that $V_0 \oplus V_1$ admits fixed point data if and only if V_1 is even-dimensional.

Exercise 6. In this exercise we consider the special case of Exercise 4 for the spin bordism category with its tangential O-action. So consider the homotopy pullback

$$\begin{array}{ccccc} B\,\mathrm{Spin}^{\mathbb{Z}/4} & \longrightarrow & & & B\,\mathrm{Spin} \\ \downarrow & & & & \downarrow \\ BO \times BO(1) & \xrightarrow{\Delta} & BO \times BO(1) \times BO(1) & \xrightarrow{\oplus} & BO \end{array} .$$

Show that a $\mathrm{Spin}^{\mathbb{Z}/4}$-structure on a vector bundle $V \to X$ consists of a line bundle $L \to X$ and a spin structure on $V \oplus L \oplus L$. Use the Whitney sum formula to conclude that a $\mathrm{Spin}^{\mathbb{Z}/4}$-structure is an orientation on V and a choice of square root of $w_2(V)$ under the cup product ring structure on $H^*(X; \mathbb{Z}/2)$.

Exercise 7. Consider the two stable tangential structures $X_1 := BO \times B\mathbb{Z}/2$ and $X_2 := B\,\mathrm{Spin}$.

a) Show that in dimension one $X_1(1) \cong X_2(1)$ are equivalent as unstable tangential structures, i.e. commuting with the map to $BO(1)$. Therefore $\mathrm{Bord}_{1,0}^{X_1} \cong \mathrm{Bord}_{1,0}^{X_2}$ as $\mathbb{E}_\infty$-categories.

b) (Difficult) If $\mathrm{ev}\colon +\sqcup- \to \emptyset$ denotes an evaluation, prove that $\mathrm{ev}^\dagger \circ \mathrm{ev}$ in $\mathrm{Bord}_{1,0}^{X_1}$ and $\mathrm{Bord}_{1,0}^{X_2}$ give the periodic circle (i.e. the mapping torus of the identity) and the antiperiodic circle (the periodic circle changed by the automorphism induced by the generator $g \in \mathbb{Z}/2 = \mathrm{Spin}(1)$) respectively. Conclude that $\mathrm{Bord}_{1,0}^{X_1} \not\cong \mathrm{Bord}_{1,0}^{X_2}$ as $\mathbb{E}_\infty$ dagger categories.