Exercises for lecture 4

Luuk Stehouwer

May 2026

Exercise 1. In this exercise you can use that $\Omega_1^{O(2)} = 0$ and $\Omega_2^{O(2)} \cong \mathbb{Z}$ by the Euler characteristic. Use the classification of invertible TQFTs to show that there is an isomorphism of groups $iTQFT_2^O \cong \mathbb{C}^\times$ given by $Z \mapsto Z(\mathbb{RP}^2)$.

Exercise 2. Consider the category $\mathrm{Vect}_{\mathbb{C}}[G]$ of G-graded vector spaces with its monoidal structure

$$(V \otimes W)_h = \bigoplus_{g_1, g_2 \in G : g_1 g_2 = h} V_{g_1} \otimes W_{g_2}$$

and canonical symmetric braiding

a) Compute $\mathrm{Pic}(\mathrm{Vect}_{\mathbb{C}}[G])$. What are π_0, π_1, and k?

b) Given a tangential structure $X(n) \to BO(n)$, compute the group of invertible TQFTs with target $\mathrm{Vect}_{\mathbb{C}}[G]$ in terms of $\Omega_{n-1}^{X(n)}$ and $\Omega_n^{X(n)}$.

Exercise 3. Let $A = \frac{\mathbb{C}[\epsilon]}{\epsilon}$. Define the linear map $\lambda \colon A \to \mathbb{C}$ by $\lambda(1) = 0$ and $\lambda(\epsilon) = 1$.

a) Prove that λ defines a Frobenius structure on A. (Hint: only need to show nondegeneracy)

b) Show that A does not admit a Hilbert space structure such that the unit map $\mathbb{C} \to A$ is the adjoint of λ

c) Conclude that the 2-dimensional oriented TQFT associated to A cannot be made unitary.

Exercise 4. In this exercise you can use that $\Omega^{O(1)} = 0$ and $\Omega_1^{O(1)} = \mathbb{Z}/2$ generated by a single point.

a) Use your knowledge about the homotopy groups of the spaces involved to show that every map $L\,\mathrm{Bord}_{1,0}^O \to \mathrm{Pic}(\mathrm{Vect}_{\mathbb{R}})$ of spaces/groupoids is nullhomotopic.

b) Use the short exact sequence classifying functors of Picard groupoids to show that symmetric monoidal functors $L\,\mathrm{Bord}_{1,0}^O \to \mathrm{Pic}(\mathrm{Vect}_{\mathbb{R}})$ are classified by $\mathbb{Z}/2$.

c) How can it be that the answers of the previous two questions were different? Conclude under the homotopy hypothesis that a map of spectra is strictly more structure than a map of spaces.

Exercise 5. Consider the symmetric monoidal category $\mathrm{Vect}_{\mathbb{R}}$. Give the exact sequence computing invertible TQFTs with target $\mathrm{Vect}_{\mathbb{R}}$ in terms of $\Omega_{n-1}^{X(n)}$ and $\Omega_n^{X(n)}$. Compare your result to Theorem 5.9 in the paper of Rovi and Schoenbauer. (Hint: also consider TQFTs valued in the subcategory of $\mathrm{Pic}(\mathrm{Vect}_{\mathbb{R}})$ on the morphisms that are ± 1)