Exercises for lecture 5

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Exercise 1. Let $\lambda \in \mathbb{C}^\times$ and let Z_λ be the Euler TQFT in some even dimension d, which assigns $Z_\lambda(M) = \lambda^{\chi(M)}$ to closed M.

a) Assume we take $X = BO$. Show that Z_λ is $\mathbb{Z}/2$-equivariant (a 'reflection TQFT') if and only if $\lambda \in \mathbb{R}$.

b) Use the classification of unitary invertible TQFTs to show that Z_λ is unitary if and only if $\lambda > 0$.

c) If $X = BSO$ and $d = 2$ show that Z_λ is $\mathbb{Z}/2$-equivariant if and only if λ is either real or purely imaginary.

d) Show that Z_λ is unitary if and only if $\lambda \in \mathbb{R}$.

e) What happens for $X = BSO$ and general even d? The answer will depend on d modulo 4.

Exercise 2. In this exercise you can use that $\Omega_1^{Spin(1)} = \mathbb{Z}_2 \times \mathbb{Z}_2$ with generators the two spin circles. Use the classification of invertible TQFTs to show that there is an isomorphism of groups $iTQFT_1^{\mathrm{Spin}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. What is the subgroup of bosonic TQFTs? What is the subgroup of unitary TQFTs? Discuss implications for the spin statistics theorem in these examples.

Exercise 3. Let $B = \mathrm{Alg}_{\mathbb{C}}$ be the 2-category of algebras, bimodules, and bimodule maps. There is an involution $(.)^{op} \colon \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Alg}_{\mathbb{C}}^{1-op}$ that takes the opposite algebra on objects, and the reversed bimodule on 1-morphisms. Consider the induced $\mathbb{Z}/2$-action on the maximal sub 2-groupoid $(\mathrm{Alg})^{\cong}$.

a) Prove that any $*$-algebra $(.)^* \colon A \to A$ defines a $\mathbb{Z}/2$-fixed point for this action.

b) Consider a 1-morphism M in $\mathrm{Alg}^{\cong}$ between $*$-algebras A, B, i.e. an invertible bimodule (Morita equivalence). Show that a structure of M preserving fixed points is equivalent to providing an B (or A)-valued Hermitian form on M. More precisely,

$$\langle x_1 | a x_2 \rangle = \langle a^* x_1 | x_2 \rangle \qquad \langle x_1 | x_2 b \rangle = \langle x_1 | x_2 \rangle b \qquad \langle x_1 | x_2 \rangle^* = \langle x_2 | x_1 \rangle. \tag{1}$$

c) (hard) Can you make sense of this as without requiring M to be invertible? (Hint: consider the composition of this action with the left/right adjoint functor $\mathrm{Hom}_A(M, A)$ on Alg. Your choice of left/right will pick out A- versus B-valued inner product)

d) (hard) Use the previous exercise to give the 2-category of $*$-algebras, Hermitian bimodules with one-sided inner product a $\dagger$ structure by adjoints of bimodule maps. Can you relate to the definition of dagger 2-category given in the lecture?