(∞, n)-CATEGORY THEORY – Lyne Moser
Exercises – Atlantic TQFT School 2025

1. Exercises for Lecture 1

Exercise 1.1. The goal of this exercise is to understand the structure of an n-category.

Let $\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathcal{C}at_{n-1}$ be an n-category. Recall that we can define composition and identity functors in $\mathcal{C}at_{n-1}$

$$c\colon \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \xleftarrow{\cong} \mathcal{C}_2 \xrightarrow{\langle 0,2\rangle^*} \mathcal{C}_1 \quad \text{and} \quad i\colon \mathcal{C}_0 \xrightarrow{!^*} \mathcal{C}_1.$$

Using the structure of the category Δ, understand why the following diagrams commute (up to homotopy) in $\mathcal{C}at_{n-1}$

$$\begin{tikzcd}
\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \arrow[r, "\mathcal{C}_1 \times_{\mathcal{C}_0} c"] \arrow[d, "c \times_{\mathcal{C}_0} \mathcal{C}_1"'] & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \arrow[d, "c"] \\
\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \arrow[r, "c"'] & \mathcal{C}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_1 \cong \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_0 \arrow[r, "\mathcal{C}_1 \times_{\mathcal{C}_0} i"] \arrow[dr, "\mathrm{id}_{\mathcal{C}_1}"'] & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \arrow[d, "c"] & \mathcal{C}_0 \times_{\mathcal{C}_0} \mathcal{C}_1 \cong \mathcal{C}_1 \arrow[l, "i \times_{\mathcal{C}_0} \mathcal{C}_1"'] \arrow[dl, "\mathrm{id}_{\mathcal{C}_1}"] \\
 & \mathcal{C}_1 &
\end{tikzcd}$$

This gives associativity and unitality for the composition c. You can also try to prove a version of the pentagon axiom using the structure of Δ.

Exercise 1.2. The goal of this exercise is to prove certain classical results about slices that will be used in the next lectures.

Let $\mathcal{A}$ be a category with all finite limits (in particular, terminal objects, products, and pullbacks) and $x \in \mathcal{A}$ be an object. Define the slice category $\mathcal{A}_{/x}$ to be the following pullback

$$\begin{tikzcd}
\mathcal{A}_{/x} \arrow[r] \arrow[d, "\pi"'] & \mathcal{F}\mathrm{un}([1], \mathcal{A}) \arrow[d, "(\langle 0\rangle^*, \langle 1\rangle^*)"] \\
{[0]} \times \mathcal{A} \arrow[r, "\{x\} \times \mathrm{id}_{\mathcal{A}}"'] & \mathcal{A} \times \mathcal{A}
\end{tikzcd}$$

(a) Prove that, for all object $p\colon a \to x$, $q\colon b \to x$ in $\mathcal{A}_{/x}$, the mapping groupoid of $\mathcal{A}_{/x}$ at p, q is given by the following pullback in $\mathcal{G}pd$

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{A}_{/x}}(a, b) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{A}}(a, b) \arrow[d, "q_*"] \\
{[0]} \arrow[r, "\{p\}"'] & \mathrm{Map}_{\mathcal{A}}(a, x)
\end{tikzcd}$$

(b) Prove that, for every morphism $f\colon x \to y$ in $\mathcal{A}$, we have an adjunction

$$f_!\colon \mathcal{A}_{/x} \rightleftarrows \mathcal{A}_{/y} :f^*$$

where $f_!$ is given by postcomposing with f and f^* by taking the pullback along f.

(c) Deduce that, for every object $x \in \mathcal{A}$, there is an adjunction

$$- \times x\colon \mathcal{A} \rightleftarrows \mathcal{A}_{/x} :\pi.$$

Exercise 1.3. Assume that we know that $\mathcal{C}at_{n-1}$ is cartesian closed. Then $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is also cartesian closed with internal homs denoted by X^A for all $A, X \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$. The goal of this exercise is to prove that $(\mathcal{C})\mathcal{S}eg(\mathcal{C}at_{n-1})$ is cartesian closed with internal homs those of $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$.

Let $X \in (\mathcal{C})\mathcal{S}eg(\mathcal{C}at_{n-1})$ be a (complete) Segal object in $\mathcal{C}at_{n-1}$.

(a) Prove that the internal hom $X^{[1]}$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is a (complete) Segal object in $\mathcal{C}at_{n-1}$.
Hint: Use that the product $[1] \times [m]$ is the colimit of the diagram

$$\begin{tikzcd}
& {[m]} \arrow[dl, "d^1"'] \arrow[dr, "d^1"] & & {[m]} \arrow[dl, "d^2"'] \arrow[dr, "d^2"] & & \cdots \arrow[dl] \arrow[dr] & & {[m]} \arrow[dl, "d^m"'] \arrow[dr, "d^m"] & \\
{[m+1]} & & {[m+1]} & & {[m+1]} & & {[m+1]} & & {[m+1]}
\end{tikzcd}$$

(b) Prove that the internal hom $X^{[m]}$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is a (complete) Segal object in $\mathcal{C}at_{n-1}$.
Hint: Write $[m]$ as a retract of the product $[1]^{\times m}$.

(c) Prove that, for every object $A \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, the internal hom X^A in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is a (complete) Segal object in $\mathcal{C}at_{n-1}$.

(d) Deduce that $(\mathcal{C})\mathcal{S}eg(\mathcal{C}at_{n-1})$ is cartesian closed with internal homs those of $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$.

Exercise 1.4. Let $\mathcal{F}in$ be the category of finite sets and maps of sets. Show that the span category $\mathcal{S}pan(\mathcal{F}in)$ is **semiadditive**, i.e., it has all finite products and coproducts and they agree.

2. EXERCISES FOR LECTURE 2

Exercise 2.1. The goal of this exercise is to prove that $\mathcal{C}at_{n-1/X}^{\Delta^{\mathrm{op}}}$ is tensored, cotensored and enriched over $\mathcal{C}at_{n-1}$.

(a) Let $L\colon \mathcal{A} \rightleftarrows \mathcal{B} : R$ be an adjunction, and let $x \in \mathcal{A}$ be an object. Prove that there is an induced adjunction

$$L\colon \mathcal{A}_{/x} \rightleftarrows \mathcal{B}_{/Lx} : (\eta_x)^* R$$

where the right adjoint is the composite $\mathcal{B}_{/Lx} \xrightarrow{R} \mathcal{A}_{/RLx} \xrightarrow{\eta_x^*} \mathcal{A}_{/x}$ with the second functor obtained by taking pullbacks along the unit component $\eta_x\colon x \to RLx$.

(b) Given $X \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, show that the functor

$$X \times \mathrm{cst}(-)\colon \mathcal{C}at_{n-1} \to \mathcal{C}at_{n-1/X}^{\Delta^{\mathrm{op}}}$$

preserves binary products.

(c) Given $X \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, show that the bifunctor

$$- \otimes -\colon \mathcal{C}at_{n-1} \times \mathcal{C}at_{n-1/X}^{\Delta^{\mathrm{op}}} \xrightarrow{\mathrm{cst}(-)\times \mathrm{id}} \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}} \times \mathcal{C}at_{n-1/X}^{\Delta^{\mathrm{op}}} \xrightarrow{-\times -} \mathcal{C}at_{n-1/X}^{\Delta^{\mathrm{op}}}.$$

is part of a two-variable adjunction with hom $(n-1)$-categories and cotensors given by the following pullbacks in $\mathcal{C}at_{n-1}$ and $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ respectively

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}at_{n-1/X}^{\Delta^{\mathrm{op}}}}(A,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_{\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}}(A,B) \arrow[d, "q_*"] \\
{[0]} \arrow[r, "p"'] & \mathrm{Hom}_{\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}}(A,X)
\end{tikzcd}
\qquad
\begin{tikzcd}
q^{\mathcal{Y}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\mathrm{cst}\,\mathcal{Y}} \arrow[d, "q_*"] \\
X \arrow[r, "!^*"'] & X^{\mathrm{cst}\,\mathcal{Y}}
\end{tikzcd}$$

for all $p\colon A \to X$, $q\colon B \to X$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ and all $\mathcal{Y} \in \mathcal{C}at_{n-1}$.
Hint: Use the tensor-cotensor structure of $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ and part (a).

(d) Prove that if $q\colon B \to X$ is a double $(n-1)$-left fibration, then the cotensor $q^{\mathcal{Y}} \to X$ is also a double $(n-1)$-left fibration, for all $\mathcal{Y} \in \mathcal{C}at_{n-1}$.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be tensored, enriched, and cotensored enriched categories over $\mathcal{C}at_{n-1}$. An n**-adjunction** between their associated n-category $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ is an adjunction $L\colon \mathcal{A} \rightleftarrows \mathcal{B} : R$ such that the left adjoint preserves tensors. We write $L\colon \underline{\mathcal{A}} \rightleftarrows \underline{\mathcal{B}} : R$.

Exercise 2.2. The goal of this exercise is to prove that n-adjunctions between tensored, enriched, and cotensored categories over $\mathcal{C}at_{n-1}$ satisfy an expected universal property with respect to hom $(n-1)$-categories.

Let $L\colon \underline{\mathcal{A}} \rightleftarrows \underline{\mathcal{B}} : R$ be an n-adjunction as above. Prove that, for all objects $a \in \mathcal{A}$ and $b \in \mathcal{B}$, there is a natural isomorphism in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{\mathcal{A}}(a, Rb) \cong \mathrm{Hom}_{\mathcal{B}}(La, b).$$

3. Exercises for Lecture 3

Definition. A map $f\colon A \to B$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is a **covariant strong deformation retract** if there is a retraction $r\colon B \to A$ such that $rf = \mathrm{id}_A$ and a map $h\colon B \times [1] \to B$ that fits in the following commutative diagram in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$.

$$\begin{tikzcd}
B \arrow[r, "\mathrm{id}_B \times \langle 0 \rangle"] \arrow[d, "r"'] & B \times [1] \arrow[d, "h"'] & B \arrow[l, "\mathrm{id}_B \times \langle 1 \rangle"'] \arrow[dl, "\mathrm{id}_B"] \\
A \arrow[r, "f"'] & B &
\end{tikzcd}$$

Exercise 3.1. The goal of this exercise is to prove that covariant strong deformation retracts are covariant equivalences over any base.

(a) Prove that a map $p\colon A \to X$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is a double $(n-1)$-left fibration if and only if the following commutative square is a pulllback square in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$.

$$\begin{tikzcd}
A^{[1]} \arrow[r, "\langle 0 \rangle^*"] \arrow[d, "p_*"'] & A \arrow[d, "p"] \\
X^{[1]} \arrow[r, "\langle 0 \rangle^*"'] & X
\end{tikzcd}$$

(b) Given maps $f\colon A \to B$ and $B \times [1] \to X$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, the map

$$g \coloneqq (f \times \mathrm{id}_{[1]}) + (\mathrm{id}_B \times \langle 0 \rangle)\colon (A \times [1]) \amalg_A B \to B \times [1]$$

is a double $(n-1)$-covariant equivalence over X.

Hint: Prove that the following commutative square is a pullback square in $\mathcal{C}at_{n-1}$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}}(B \times [1], C) \arrow[r, "g^*"] \arrow[d, "p_*"'] & \mathrm{Hom}_{\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}}((A \times [1]) \amalg_A B, C) \arrow[d, "p_*"] \\
\mathrm{Hom}_{\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}}(B \times [1], X) \arrow[r, "g^*"'] & \mathrm{Hom}_{\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}}((A \times [1]) \amalg_A B, X)
\end{tikzcd}$$

and use Exercise 2.1 (c).

(c) Prove that, if $f\colon A \to B$ is a covariant strong deformation retract in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ and $B \to X$ is a map in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, then f is a double $(n-1)$-covariant equivalence over X.

Hint: Prove that f is a retract of a map as in (b).

Exercise 3.2. Prove the following key lemma used in the proof of the Yoneda lemma.

Given an n-category $\mathcal{C}$ and a functor $F\colon \mathcal{D} \to \mathcal{E}$ in $\mathcal{C}at_n$, consider a commutative square in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ as below left, where p and q are double $(n-1)$-left fibrations.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "\mathrm{id}_{\mathcal{C}} \times F"'] & \mathcal{C} \times \mathcal{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{x\} \times_{\mathcal{C}} A \arrow[r, "\{x\} \times_{\mathcal{C}} f"] \arrow[d, "\{x\} \times_{\mathcal{C}} p"'] & \{x\} \times_{\mathcal{C}} B \arrow[d, "\{x\} \times_{\mathcal{C}} q"] \\
\mathcal{D} \arrow[r, "F"'] & \mathcal{E}
\end{tikzcd}$$

Prove that f is a double $(n-1)$-covariant equivalence over $\mathcal{C} \times \mathcal{E}$ if and only if, for every object $x \in \mathcal{C}$, the map $\{x\} \times_{\mathcal{C}} f$ (as above right) is a double $(n-1)$-covariant equivalence over $\mathcal{E}$. **Hint:** Use Quillen's Theorem A and that taking the pullback along a double $(n-1)$-left fibrations over an n-category preserves double $(n-1)$-contravariant equivalences.

4. EXERCISES FOR LECTURE 4

Exercise 4.1. The equivalence of (i) and (ii) in (a) and the result in (b) were applied to the n-Yoneda functor $y_{\mathcal{C}}\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at_{n-1}]$ to prove that it is fully faithful and the n-Yoneda Lemma.

(a) Prove that the following are equivalent for a functor $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$:
   (i) it is fully faithful;
   (ii) the following square is a pullback square in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$;

$$\begin{tikzcd}
\mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[r, "\mathbb{T}\mathrm{w}(F)"] \arrow[d] & \mathbb{T}\mathrm{w}(\mathcal{D}) \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "F^{\mathrm{op}} \times F"'] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D}
\end{tikzcd}$$

   (iii) the functor $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is fully faithful;
   (iv) the functor $F_!\colon [\mathcal{C}, \underline{\mathcal{C}at}_{n-1}] \to [\mathcal{D}, \underline{\mathcal{C}at}_{n-1}]$ is fully faithful, where $F_!$ is the composite

$$[\mathcal{C}, \underline{\mathcal{C}at}_{n-1}] \cong \underline{\mathcal{D}bl\mathcal{LF}ib}(\mathcal{C}) \hookrightarrow \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}} \xrightarrow{F_!} \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{D}} \xrightarrow{L^{\mathcal{L}}_{\mathcal{D}}} \underline{\mathcal{D}bl\mathcal{LF}ib}(\mathcal{D}) \cong [\mathcal{D}, \underline{\mathcal{C}at}_{n-1}].$$

(b) Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$, define u_F to be the unique map given by the universal property of pullbacks making the following diagram commute in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, and prove that it is a double $(n-1)$-covariant equivalence over $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$. **Hint:** Use Exercise 3.2.

$$\begin{tikzcd}
\mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[r, dashed, "u_F"] \arrow[d] \arrow[rr, bend left, "\mathbb{T}\mathrm{w}(F)"] & \int_{\mathcal{C}^{\mathrm{op}} \times \mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(F(-), -) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{T}\mathrm{w}(\mathcal{D}) \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \times F"'] & \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "F^{\mathrm{op}} \times \mathrm{id}_{\mathcal{D}}"'] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D}
\end{tikzcd}$$

Exercise 4.2. Given a functor $F\colon \mathcal{J} \to \mathcal{C}$ in $\mathcal{C}at_n$ and an object $j \in \mathcal{J}$, prove that

(a) the $\mathrm{Hom}_{\mathcal{J}}(j, -)$-weighted n-limit of F exists, and can be computed as

$$\lim_{\mathcal{J}}^{\mathrm{Hom}_{\mathcal{J}}(j,-)} F \cong Fj.$$

(b) the $\mathrm{Hom}_{\mathcal{J}}(-, j)$-weighted n-colimit of F exists, and can be computed as

$$\mathrm{colim}_{\mathcal{J}}^{\mathrm{Hom}_{\mathcal{J}}(-,j)} F \cong Fj.$$