(∞, n)-CATEGORY THEORY
Course at Atlantic TQFT School 2025

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Contents

Acknowledgment. The material in these notes is based on joint work with Rasekh and Rovelli available on my website lynemoser.com.

Remark. Readers who prefer to avoid (∞, n)-categories for $n > 1$, or the homotopical aspects of higher category theory, may safely restrict to $n = 1$ or to strict/weak n-categories. The arguments presented here are formal and so also make sense in these stricter contexts.

1. Lecture 1: First definitions and examples

1.1. **Intuitive definitions and examples.** We begin with an intuitive definition of an (∞, n)-category, together with some motivating examples of such structures.

"Definition". An (∞, n)**-category** is a structure that has

$$
\begin{array}{l}
\triangleright\ \text{objects}\ \bullet \\
\left.\begin{array}{l}
\triangleright\ \text{1-morphisms}\ \bullet \longrightarrow \bullet \\
\triangleright\ \text{2-morphisms}\ \bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} \bullet \\
\triangleright\ \dots \\
\triangleright\ n\text{-morphisms} \\
\left.\begin{array}{l}
\triangleright\ (n+1)\text{-morphisms} \\
\triangleright\ \dots
\end{array}\right\}\text{invertible}
\end{array}\right\}(\infty, n-1)\text{-category}
\end{array}
$$

coming with identities and compositions that are associative and unital up to (coherent) higher invertible morphisms.

Example 1.1.1. **Homotopy hypothesis:** Topological spaces (up to weak homotopy equivalences) are precisely the ∞-groupoids, i.e., the $(\infty, 0)$-categories.

Given a topological space X, its associated ∞-groupoid has:

- $\triangleright$ objects: points in X
- $\triangleright$ 1-morphisms: paths/homotopies between points in X
- $\triangleright$ 2-morphisms: homotopies between paths in X

- 3-morphisms: homotopies between homotopies between paths in X
- ...

Compositions are given by concatenations of homotopies and identities by constant homotopies.

Example 1.1.2. The (∞, n)-category of spans of finite sets has:

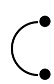
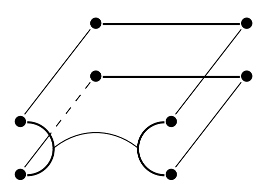
- objects: finite sets $S, T, \ldots$
- 1-morphisms: spans $S \leftarrow M \rightarrow T$
- 2-morphisms: spans between spans $\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ S & P \arrow[u] \arrow[d] & T \\ & N \arrow[ul] \arrow[ur] & \end{tikzcd}$
- ...
- n-morphisms: spans between spans ... between spans
- $(n+1)$-morphisms: isomorphisms of spans $\begin{tikzcd} & M \arrow[dl] \arrow[dr] \arrow[dd, "\cong"] & \\ S & & T \\ & N \arrow[ul] \arrow[ur] & \end{tikzcd}$

Compositions are given by pullbacks and identities by identity spans.

Remark 1.1.3.

- This construction is not specific to finite sets; one can form in a similar way the (∞, n)-category of spans in any category or $(\infty, 1)$-category with pullbacks; see [Haugseng].
- The (∞, n)-categories of spans are fully dualizable symmetric monoidal (∞, n)-categories, and so we can build extended n-TQFTs into these (∞, n)-categories [Haugseng].

Example 1.1.4. The (∞, n)-category of cobordisms has:

- objects: 0-manifolds $\bullet$, ...
- 1-morphisms: 1-cobordisms  , 
- 2-morphisms: 2-cobordisms with corners 
- ...
- n-morphisms: n-cobordisms with corners
- $(n+1)$-morphisms: homeomorphisms (of n-manifolds)
- $(n+2)$-morphisms: smooth homotopies between homeomorphisms
- ...

Compositions are given by gluings and identities by products with the closed interval.

Remark 1.1.5.

- The (∞, n)-category of cobordisms is at the heart of TQFT. An extended n-TQFT is a symmetric monoidal functor out of the (∞, n)-category of cobordisms (with framing).
- **Cobordism hypothesis:** The (∞, n)-category of cobordisms (with framing) is the free fully dualizable symmetric monoidal (∞, n)-category on a point.

1.2. ∞-Category theory. Next, we introduce the required ∞-category theory for these notes. This provides the foundations in which we formulate the theory of (∞, n)-categories. From now on, we will drop the "∞" from the notation and terminology and refer to (∞, n)-categories simply as n-categories. We reserve the term (∞-)category for the objects of the ambient theory, in contrast to $(\infty, 1)$- (or simply 1-)categories.

Notation 1.2.1. We denote by

▷ $\mathcal{C}at$ the (∞-)category of (∞-)categories;
▷ $\mathcal{G}pd$ the (∞-)category of (∞-)groupoids.

Remark 1.2.2. There are several ways of making precise what these categories are. Some are based on set theoretical foundations:

▷ model categories:
  – Kan complexes (up to weak homotopy equivalences) [Quillen];
  – Quasi-categories (up to categorical equivalences) [Joyal, Lurie];
  – Complete Segal spaces (up to Dwyer-Kan equivalences) [Rezk, Rasekh].
▷ ∞-cosmoi [Riehl–Verity].

Some are based on homotopy type theoretical foundations (in which sets are replaces by ∞-groupoids):

▷ simplicial homotopy type theory [Riehl–Shulman, Buchholtz–Gratzer–Weinberger];
▷ synthetic category theory [Cisinski–Cnossen–Nguyen–Walde]

The latter is developing all of ∞-category theory from a minimal set of axioms.

The goal of these notes is to explain how to develop (∞, n)-category theory from a purely formal point of view, so that definitions, statements, and proofs make sense in all of the above setting, in particular in formal contexts such as the synthetic one.

Fact 1. A category $\mathcal{A}$ has:

▷ an **underlying groupoid** $\mathcal{A}^{\simeq}$; the points of $\mathcal{A}^{\simeq}$ are the **objects** of $\mathcal{A}$;
▷ for all objects $x, y \in \mathcal{A}$, a **mapping groupoid** $\mathrm{Map}_{\mathcal{A}}(x, y)$ of morphisms; the points of $\mathrm{Map}_{\mathcal{A}}(x, y)$ are the **morphisms** of $\mathcal{A}$ from x to y;
▷ there is a functor $\mathrm{Map}_{\mathcal{A}}(-,-)\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{G}pd$; so there is a composition for morphisms.

Definition 1.2.3. A square in a category $\mathcal{A}$ as below **commutes** if there is a homotopy in $\mathrm{Map}_{\mathcal{A}}(x, y')$ between the composites hf to $f'g$.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ x' \arrow[r, "f'"'] & y' \end{tikzcd}$$

Commutativity is *not* a property of a diagram, but is itself data, namely the data of a homotopy.

Definition 1.2.4. An **isomorphism** in a category $\mathcal{A}$ is a morphism $f\colon x \to y$ in $\mathcal{A}$ such that there exists a morphism $g\colon y \to x$ in $\mathcal{A}$ making the following triangles commute.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "\mathrm{id}_x"'] & y \arrow[d, "g"] \\ & x \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "g"] \arrow[dr, "\mathrm{id}_y"'] & x \arrow[d, "f"] \\ & y \end{tikzcd}$$

Fact 2. For all categories $\mathcal{A}$ and $\mathcal{B}$, there is a category $\mathcal{F}\mathrm{un}(\mathcal{A},\mathcal{B})$ of functors $\mathcal{A} \to \mathcal{B}$ and natural transformations. These assemble into a functor $\mathcal{F}\mathrm{un}(-,-)\colon \mathcal{C}at^{\mathrm{op}} \times \mathcal{C}at \to \mathcal{C}at$.

Definition 1.2.5. A functor $F\colon \mathcal{A} \to \mathcal{B}$ in $\mathcal{C}at$ is **fully faithful** if, for all objects $x, y \in \mathcal{A}$,

$$\mathrm{Map}_{\mathcal{A}}(x,y) \to \mathrm{Map}_{\mathcal{B}}(Fx,Fy)$$

is an isomorphism in $\mathcal{G}pd$.

Definition 1.2.6. An **adjunction** is the data of

- ▷ functors $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$;
- ▷ natural transformations $\eta\colon \mathrm{id}_{\mathcal{A}} \Rightarrow RL$ and $\varepsilon\colon LR \Rightarrow \mathrm{id}_{\mathcal{B}}$;

satifsfying the triangle identities. We denote such an adjunction by $L\colon \mathcal{A} \rightleftarrows \mathcal{B} : R$ or $L \dashv R$.

Theorem 1.2.7. *A pair $L\colon \mathcal{A} \rightleftarrows \mathcal{B} : R$ is an adjunction if and only if, for all objects $a \in \mathcal{A}$ and $b \in \mathcal{B}$, there is a natural isomorphism in $\mathcal{G}pd$*

$$\mathrm{Map}_{\mathcal{A}}(a, Rb) \cong \mathrm{Map}_{\mathcal{B}}(La, b).$$

Definition 1.2.8. A category $\mathcal{A}$ **has all (co)limits** if, for every $\mathcal{J} \in \mathcal{C}at$, the constant functor $\mathrm{cst}\colon \mathcal{A} \to \mathcal{F}\mathrm{un}(\mathcal{J},\mathcal{A})$ has a right adjoint $\lim_{\mathcal{J}}\colon \mathcal{F}\mathrm{un}(\mathcal{J},\mathcal{A}) \to \mathcal{A}$ (resp. a left adjoint $\mathrm{colim}_{\mathcal{J}}\colon \mathcal{F}\mathrm{un}(\mathcal{J},\mathcal{A}) \to \mathcal{A}$).

Remark 1.2.9. A category $\mathcal{A}$ has all limits if and only if, for all functors $F\colon \mathcal{J} \to \mathcal{A}$ and all objects $x \in \mathcal{A}$, there is a natural isomorphism in $\mathcal{G}pd$

$$\mathrm{Map}_{\mathcal{A}}(x, \lim_{\mathcal{J}} F) \cong \mathrm{Map}_{\mathcal{F}\mathrm{un}(\mathcal{J},\mathcal{A})}(\mathrm{cst}\, x, F).$$

Fact 3. The categories $\mathcal{G}pd$ and $\mathcal{C}at$ have all (small) (co)limits.

Definition 1.2.10. Given a category $\mathcal{A}$ with all (small) (co)limits, we call **canonical functor** the unique colimit-preserving functor $\mathcal{G}pd \to \mathcal{A}$ that preserves the terminal object.

1.3. **Defining (∞, n)-categories.** We are now ready to introduce (∞, n)-categories.

Definition 1.3.1. Let Δ be the category with:

- ▷ objects: finite ordered sets $[m] \coloneqq \{0 < 1 < \ldots < m\}$ for $m \geq 0$ an integer;
- ▷ morphisms: order-preserving maps.

Notation 1.3.2. Given a map $\alpha\colon [m] \to [n]$ in Δ, we denote it by its image $\alpha = \langle \alpha(0), \ldots, \alpha(m) \rangle$. For example, the map $\langle 0, 2\rangle\colon [1] \to [2]$ sends $0 \mapsto 0$ and $1 \mapsto 2$. The object $[0] \in \Delta$ is terminal and we denote by $!\colon [m] \to [0]$ the unique map, for all $m \geq 1$.

Notation 1.3.3. Given a category $\mathcal{A}$,

- ▷ we write $\mathcal{A}^{\Delta^{\mathrm{op}}} \coloneqq \mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathcal{A})$ for the category of **simplicial objects** in $\mathcal{A}$;
- ▷ given a functor $X\colon \Delta^{\mathrm{op}} \to \mathcal{A}$, we write $X_m \coloneqq X([m])$ for $m \geq 0$, and $\alpha^* \coloneqq X(\alpha)\colon X_n \to X_m$ for $\alpha\colon [m] \to [n]$ in Δ.

Definition 1.3.4. Let $\mathcal{A}$ be a category with all (co)limits such that the canonical functor $\mathcal{G}pd \to \mathcal{A}$ is fully faithful. A functor $X\colon \Delta^{\mathrm{op}} \to \mathcal{A}$ is

- ▷ **Segal** if the map
$$(\langle 0,1\rangle^*, \langle 1,2\rangle^*, \ldots \langle m-1, m\rangle^*)\colon X_m \to X_1 \times_{X_0} X_1 \times_{X_0} \ldots \times_{X_0} X_1$$
is an isomorphism in $\mathcal{A}$, for all $m \geq 2$;
- ▷ **complete** if the map
$$X_0 \to X_0 \times_{X_1} X_3 \times_{X_1} X_0$$
is an isomorphism in $\mathcal{A}$, where the right-hand side is the limit of the diagram

$$\begin{tikzcd}[column sep=small] X_0 \arrow[dr, "!^*"'] & & X_3 \arrow[dl, "\langle 0,2\rangle^*"] \arrow[dr, "\langle 1,3\rangle^*"] & & X_0 \arrow[dl, "!^*"] \\ & X_1 & & X_1 & \end{tikzcd}$$

- ▷ **globular** if X_0 is in the image of the canonical functor $\mathcal{G}pd \to \mathcal{A}$.

We denote by $\mathcal{S}eg(\mathcal{A})$ (resp. $\mathcal{CS}eg(\mathcal{A})$, resp. $\mathcal{C}at(\mathcal{A})$) the full subcategory of $\mathcal{A}^{\Delta^{\mathrm{op}}}$ spanned by the Segal objects (resp. complete Segal objects, resp. complete Segal globular objects, also called **category objects**) in $\mathcal{A}$.

Proposition 1.3.5. *Let $\mathcal{A}$ be as above. The categories $\mathcal{S}eg(\mathcal{A})$, $\mathcal{CS}eg(\mathcal{A})$ and $\mathcal{C}at(\mathcal{A})$ also have all (co)limits and the canonical functor from $\mathcal{G}pd$ is fully faithful.*

Definition 1.3.6. We define the category $\mathcal{C}at_n$ of n-**categories** by induction on $n \geq 0$.

- ▷ For $n = 0$, we set $\mathcal{C}at_0 \coloneqq \mathcal{G}pd$.
- ▷ For $n \geq 1$, we set $\mathcal{C}at_n \coloneqq \mathcal{C}at(\mathcal{C}at_{n-1})$.

When $n = 1$, we have an isomorphism of categories $\mathcal{C}at \cong \mathcal{C}at_1$.

Definition 1.3.7. Let $\mathcal{C}$ be an n-category. An **object** of $\mathcal{C}$ is a point $x \in \mathcal{C}_0$. Given two objects $x, y \in \mathcal{C}$, the **hom** $(n-1)$**-category** $\mathrm{Hom}_{\mathcal{C}}(x,y)$ between x and y is the pullback in $\mathcal{C}at_{n-1}$

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_1 \arrow[d, "(\langle 0\rangle^*, \langle 1\rangle^*)"] \\ {[0]} \arrow[r, "(x,y)"'] & \mathcal{C}_0 \times \mathcal{C}_0 \end{tikzcd}$$

Remark 1.3.8. Let us briefly interpret the definition. Let $\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathcal{C}at_{n-1}$ be an n-category.

- ▷ The $(n-1)$-categories $\mathcal{C}_0$ and $\mathcal{C}_1$ are the $(n-1)$-categories of objects and morphisms.
- ▷ The Segal conditions allow us to define a composition
$$\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \xleftarrow{\cong} \mathcal{C}_2 \xrightarrow{\langle 0,2\rangle^*} \mathcal{C}_1$$
and encode all coherences for associativity and unitality of composition (Exercise).
- ▷ The globular and completeness conditions tell us that $\mathcal{C}_0$ is in fact a groupoid and is identified with the underlying groupoid of $\mathcal{C}$. Indeed, a point in $\mathcal{C}_0 \times_{\mathcal{C}_1} \mathcal{C}_3 \times_{\mathcal{C}_1} \mathcal{C}_0$ is the data of an invertible morphism in $\mathcal{C}$.

Definition 1.3.9. A functor $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$ is **fully faithful** if, for all objects $x, y \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$$

is an isomorphism in $\mathcal{C}at_{n-1}$.

Proposition 1.3.10. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an isomorphism in $\mathcal{C}at_n$ if and only if the induced map $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ is an isomorphism in $\mathcal{G}pd$ and F is fully faithful.*

Theorem 1.3.11. *The category $\mathcal{C}at_n$ is cartesian closed, i.e., for every n-category $\mathcal{J}$, there is an adjunction $- \times \mathcal{J}\colon \mathcal{C}at_n \rightleftarrows \mathcal{C}at_n : [\mathcal{J}, -]$.*

Proof idea. We prove the result by induction on $n \geq 0$. The case $n = 0$ is clear. Assume that $n > 0$ and that $\mathcal{C}at_{n-1}$ is cartesian closed. Then $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is also cartesian closed, and we denote by X^A its internal homs for all $A, X \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$. Moreover, the categories $\mathcal{S}eg(\mathcal{C}at_{n-1})$ and $\mathcal{CS}eg(\mathcal{C}at_{n-1})$ are also cartesian closed, with internal homs those of $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ (Exercise).

Finally, the inclusion $\mathcal{C}at_n \hookrightarrow \mathcal{CS}eg(\mathcal{C}at_{n-1})$ preserves binary products and admits a right adjoint $U\colon \mathcal{CS}eg(\mathcal{C}at_{n-1}) \to \mathcal{C}at_n$. Hence $\mathcal{C}at_n$ is also cartesian closed with internal homs given by $[\mathcal{J}, \mathcal{C}] \coloneqq U(\mathcal{C}^{\mathcal{J}})$ for all $\mathcal{J}, \mathcal{C} \in \mathcal{C}at_n$. ∎

2. Lecture 2: Examples of (∞, n)-categories and (un)straightening

2.1. **Tensored, enriched, and cotensored categories.** The category $\mathcal{C}at_{n-1}$ is cartesian closed, and so has hom $(n-1)$-categories given by its internal homs. This should determine an n-category. We describe here a general construction that produces n-categories from similar data, and all the ambient n-categories in these notes arise in this way. We only streamline the construction, and refer the readers to [MRR] for a complete proof.

"Definition". A **tensored category over** $\mathcal{C}at_{n-1}$ is a category $\mathcal{A}$ which has the structure of a $\mathcal{C}at_{n-1}$-algebra; in particular, there is a **tensor**

$$- \otimes_{\mathcal{A}} - \colon \mathcal{A} \times \mathcal{C}at_{n-1} \to \mathcal{A}$$

and, for all objects $\mathcal{X}, \mathcal{Y} \in \mathcal{C}at_{n-1}$, there is a natural isomorphism of functors $\mathcal{A} \to \mathcal{A}$

$$(- \otimes_{\mathcal{A}} \mathcal{X}) \otimes_{\mathcal{A}} \mathcal{Y} \cong - \otimes_{\mathcal{A}} (\mathcal{X} \times \mathcal{Y});$$

but it also has a lot more higher coherence data. It is further **enriched and cotensored** if the tensor is part of a two-variable adjunction, i.e., for all objects $a \in \mathcal{A}$ and $\mathcal{X} \in \mathcal{C}at_{n-1}$, we have adjunctions

$$a \otimes_{\mathcal{A}} (-) \colon \mathcal{C}at_{n-1} \rightleftarrows \mathcal{A} : \mathrm{Hom}_{\mathcal{A}}(a, -) \quad \text{and} \quad (-) \otimes_{\mathcal{A}} \mathcal{X} \colon \mathcal{A} \rightleftarrows \mathcal{C}at_{n-1} : (-)^{\mathcal{X}}.$$

In particualr, there are induced functors

$$\mathrm{Hom}_{\mathcal{A}}(-,-) \colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}at_{n-1} \quad \text{and} \quad (-)^{(-)} \colon \mathcal{C}at_{n-1}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{A}.$$

Remark 2.1.1. The idea is that every such tensored, enriched, and cotensored category over $\mathcal{C}at_{n-1}$ gives rise to an n-category whose

- ▷ underlying groupoid is $\mathcal{A}^{\simeq}$;
- ▷ hom $(n-1)$-categories are $\mathrm{Hom}_{\mathcal{A}}(a, b)$ for all objects $a, b \in \mathcal{A}$.

Given objects $a, b, c \in \mathcal{A}$, the composition functor $\mathrm{Hom}_{\mathcal{A}}(a, b) \times \mathrm{Hom}_{\mathcal{A}}(b, c) \to \mathrm{Hom}_{\mathcal{A}}(a, c)$ is determined by Yoneda using that, for all $\mathcal{X} \in \mathcal{C}at_{n-1}$, the induced map

$$\mathrm{Map}_{\mathcal{C}at_{n-1}}(\mathcal{X}, \mathrm{Hom}_{\mathcal{A}}(a, b) \times \mathrm{Hom}_{\mathcal{A}}(b, c)) \to \mathrm{Map}_{\mathcal{C}at_{n-1}}(\mathcal{X}, \mathrm{Hom}_{\mathcal{A}}(a, c))$$

corresponds via the adjunction to the map

$$\mathrm{Map}_{\mathcal{A}}(a \otimes_{\mathcal{A}} \mathcal{X}, b) \times \mathrm{Map}_{\mathcal{A}}(b \otimes_{\mathcal{A}} \mathcal{X}, c) \to \mathrm{Map}_{\mathcal{A}}(a \otimes_{\mathcal{A}} \mathcal{X}, c)$$

which sends a pair $(f \colon a \otimes_{\mathcal{A}} \mathcal{X} \to b, g \colon b \otimes_{\mathcal{A}} \mathcal{X} \to c)$ to the composite

$$a \otimes_{\mathcal{A}} \mathcal{X} \xrightarrow{\mathrm{id}_a \otimes_{\mathcal{A}} \mathrm{diag}} a \otimes_{\mathcal{A}} (\mathcal{X} \times \mathcal{X}) \cong (a \otimes_{\mathcal{A}} \mathcal{X}) \otimes_{\mathcal{A}} \mathcal{X} \xrightarrow{f \otimes_{\mathcal{A}} \mathcal{X}} b \otimes_{\mathcal{A}} \mathcal{X} \xrightarrow{g} c.$$

To provide the data of such a category $\mathcal{A}$, we need to provide a full tower of coherences. In order to avoid this, we restrict ourselves to categories $\mathcal{A}$ where the tensor structure is of a specific form, and this will reduce the verification of compatibilities to binary operations only, which is very convenient in practice.

"Definition". We construct a category $\mathcal{C}at^{\otimes}$ whose:

- ▷ objects: functors $\iota_{\mathcal{A}} \colon \mathcal{C}at_{n-1} \to \mathcal{A}$ in $\mathcal{C}at$ such that $\mathcal{A}$ has all finite limits and $\iota_{\mathcal{A}}$ preserves binary products; we get a tensor

  $$- \otimes_{\mathcal{A}} - \colon \mathcal{C}at_{n-1} \times \mathcal{A} \xrightarrow{\iota_{\mathcal{A}} \times \mathrm{id}_{\mathcal{A}}} \mathcal{A} \times \mathcal{A} \xrightarrow{- \times -} \mathcal{A}$$

  and we require that it is part of a two-variable adjunction;

- morphisms: functors $\mathcal{A} \to \mathcal{B}$ together with a natural transformation $\alpha_F \colon F\iota_{\mathcal{A}} \Rightarrow \iota_{\mathcal{B}}$ of functors $\mathcal{C}at_{n-1} \to \mathcal{B}$ such that, for every $\mathcal{X} \in \mathcal{C}at_{n-1}$ and $a \in \mathcal{A}$, the induced map
$$F(a \otimes_{\mathcal{A}} \mathcal{X}) = F(a \times \iota_{\mathcal{A}} X) \to Fa \times F\iota_{\mathcal{A}} X \xrightarrow{\mathrm{id}_{Fa} \times (\alpha_F)_{\mathcal{X}}} Fa \times \iota_{\mathcal{B}}\mathcal{X} = Fa \otimes_{\mathcal{B}} \mathcal{X}$$
is an isomorphism in $\mathcal{B}$; we say that F **preserves tensors**.

Example 2.1.2. Given objects $(\mathcal{A}, \iota_{\mathcal{A}}), (\mathcal{B}, \iota_{\mathcal{B}})$ in $\mathcal{C}at^{\otimes}$, if a functor $F \colon \mathcal{A} \to \mathcal{B}$ preserves binary products and commutes with the functors $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ from $\mathcal{C}at_{n-1}$, it preserves tensors.

Theorem 2.1.3. *There is a functor* $\underline{(-)} \colon \mathcal{C}at^{\otimes} \to \mathcal{C}at_n$.

Proof idea. The simplification here is as follows. Given objects $\mathcal{A} \in \mathcal{C}at^{\otimes}$ and $\mathcal{X} \in \mathcal{C}at_{n-1}$, we can obtain a category $\mathcal{A}_{\mathcal{X}}$ with
- underlying groupoid $\mathcal{A}^{\simeq}$;
- mapping groupoids $\mathrm{Map}_{\mathcal{A}_{\mathcal{X}}}(a, b) \coloneqq \mathrm{Map}_{\mathcal{A}}(a \otimes_{\mathcal{A}} \mathcal{X}, b)$ for all objects $a, b \in \mathcal{A}$;

as the full subcategory $\mathcal{A}_{\mathcal{X}}$ of the slice $\mathcal{A}_{/\iota_{\mathcal{A}}\mathcal{X}}$ spanned by the image of the functor
$$- \otimes_{\mathcal{A}} \mathcal{X} = - \times \iota_{\mathcal{A}}\mathcal{X} \colon \mathcal{A} \to \mathcal{A}_{/\iota_{\mathcal{A}}\mathcal{X}}.$$
Indeed, using the adjunction $- \otimes_{\mathcal{A}} \mathcal{X} = - \times \iota_{\mathcal{A}}\mathcal{X} \colon \mathcal{A} \rightleftarrows \mathcal{A}_{/\iota_{\mathcal{A}}\mathcal{X}} \colon \pi$ where π is the canonical projection, we have natural isomorphisms in $\mathcal{G}pd$
$$\mathrm{Map}_{\mathcal{A}_{\mathcal{X}}}(a, b) \cong \mathrm{Map}_{\mathcal{A}_{/\iota_{\mathcal{A}}\mathcal{X}}}(a \otimes_{\mathcal{A}} \mathcal{X}, b \otimes_{\mathcal{A}} \mathcal{X}) \cong \mathrm{Map}_{\mathcal{A}}(a \otimes_{\mathcal{A}} \mathcal{X}, b).$$
This then has composition as described in the remark above.

Given a morphism $F \colon \mathcal{A} \to \mathcal{B}$ in $\mathcal{C}at^{\otimes}$, the induced functor $F \colon \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ in $\mathcal{C}at_n$ acts as
- $F^{\simeq} \colon \mathcal{A}^{\simeq} \to \mathcal{B}^{\simeq}$ on underlying groupoids;
- for all objects $a, b \in \mathcal{A}$, the functor $\mathrm{Hom}_{\mathcal{A}}(a, b) \to \mathrm{Hom}_{\mathcal{B}}(Fa, Fb)$ is determined by Yoneda using that, for all $\mathcal{X} \in \mathcal{C}at_{n-1}$, the induced map
$$\mathrm{Map}_{\mathcal{C}at_{n-1}}(\mathcal{X}, \mathrm{Hom}_{\mathcal{A}}(a, b)) \to \mathrm{Map}_{\mathcal{C}at_{n-1}}(\mathcal{X}, \mathrm{Hom}_{\mathcal{B}}(Fa, Fb))$$
corresponds via the adjunction to the map
$$\mathrm{Map}_{\mathcal{A}}(a \otimes_{\mathcal{A}} \mathcal{X}, b) \to \mathrm{Map}_{\mathcal{B}}(F(a \otimes_{\mathcal{A}} \mathcal{X}), Fb) \cong \mathrm{Map}_{\mathcal{A}}(Fa \otimes_{\mathcal{B}} \mathcal{X}, Fb)$$
where the last isomorphism holds since F preserves tensors. □

2.2. **Examples of (∞, n)-categories and functors.** We can now applied this result to construct different n-categories and functors between them of interest.

Example 2.2.1. We have the following n-categories.
- **The n-category of $(n-1)$-categories:** The category $\mathcal{C}at_{n-1}$ with the identity functor
$$\mathrm{id}_{\mathcal{C}at_{n-1}} \colon \mathcal{C}at_{n-1} \to \mathcal{C}at_{n-1}$$
determines an object of $\mathcal{C}at^{\otimes}$ since $\mathcal{C}at_{n-1}$ is cartesian closed:
  - tensors are given by cartesian products;
  - hom $(n-1)$-categories and cotensors are both given by the internal homs.

  By the Theorem, we get an n-category $\underline{\mathcal{C}at}_{n-1}$.
- **The n-category of simplicial objects in $(n-1)$-categories:** The category $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ with the constant functor (induced by precomposing with the unique functor $\Delta^{\mathrm{op}} \to [0]$)
$$\mathrm{cst} \colon \mathcal{C}at_{n-1} \to \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$$
determines an object of $\mathcal{C}at^{\otimes}$ since cst preserves binary products, $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is cartesian closed with internal homs denoted by B^A for all $A, B \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ and $\mathrm{cst} \dashv (-)_0$: for all $A, B \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ and $\mathcal{X} \in \mathcal{C}at_{n-1}$,

  - the tensor $A \otimes \mathcal{X}$ is given by $A \times \mathrm{cst}\, \mathcal{X}$;
  - the hom $(n-1)$-category $\mathrm{Hom}_{\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}}(A,B)$ is given by $(B^A)_0$;
  - the cotensor $B^{\mathcal{X}}$ is given by $B^{\mathrm{cst}\,\mathcal{X}}$.

  By the Theorem, we get an n-category $\underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1}$.
- ▷ **The slice n-category over a simplicial object in $(n-1)$-categories:** Given an object $X \in \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$, the category $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X}$ with the functor

$$X \times \mathrm{cst}(-) \colon \mathcal{C}at_{n-1} \to \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X}$$

  determines an object of $\mathcal{C}at^{\otimes}$ (Exercise): for all maps $p\colon A \to X$ in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$ and all $\mathcal{Y} \in \mathcal{C}at_{n-1}$,
  - the tensor $p \otimes \mathcal{Y}$ is given by $A \times \mathrm{cst}\,\mathcal{Y} \xrightarrow{\pi_1} A \xrightarrow{p} X$;
  - the computation of hom $(n-1)$-categories and cotensors is left as an exercise.

  By the Theorem, we get an n-category $\underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/X}$.

Example 2.2.2. We construct functors between the above n-categories.

- ▷ Given a map $f\colon X \to Y$ in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$, the pullback functor $f^*\colon \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/Y} \to \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X}$ preserves binary products and commutes with the functors from $\mathcal{C}at_{n-1}$. So we get a functor in $\mathcal{C}at_n$

$$f^*\colon \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/Y} \to \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/X}.$$

- ▷ Given a map $f\colon X \to Y$ in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$, the postcomposition functor $f_!\colon \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X} \to \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/Y}$ comes with a natural transformation whose component at $\mathcal{Z} \in \mathcal{C}at_{n-1}$ is as below right

$$\begin{tikzcd}
& \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X} \arrow[dd, "f_!"] \\
\mathcal{C}at_{n-1} \arrow[ur, "X \times \mathrm{cst}(-)"] \arrow[dr, "Y \times \mathrm{cst}(-)"'] \arrow[r, phantom, "f \times \mathrm{cst}(-) \Swarrow"] & \\
& \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times \mathrm{cst}\,\mathcal{Z} \arrow[r, "f \times \mathrm{cst}\,\mathcal{Z}"] \arrow[d, "\pi_1"'] & Y \times \mathrm{cst}\,\mathcal{Z} \arrow[d, "\pi_1"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

  For all $p\colon A \to X$ in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$ and all $\mathcal{Z} \in \mathcal{C}at_{n-1}$, we have a canonical isomorphism over Y

$$f_!(A \times \mathrm{cst}\,\mathcal{Z}) \cong f_!(A) \times \mathrm{cst}\,\mathcal{Z}$$

  (both sides are given by $A \times \mathrm{cst}\,\mathcal{Z} \xrightarrow{\pi_1} A \xrightarrow{p} X \xrightarrow{f} Y$). Hence we get a functor in $\mathcal{C}at_n$

$$f_!\colon \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/X} \to \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/Y}.$$

Definition 2.2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be objects in $\mathcal{C}at^{\otimes}$. An n**-adjunction** between their associated n-categories $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$ is an adjunction $L\colon \mathcal{A} \rightleftarrows \mathcal{B} \colon R$ in $\mathcal{C}at$ such that the left adjoint L preserves tensors (or equivalently, the right adjoint R preserves cotensors). We write $L\colon \underline{\mathcal{A}} \rightleftarrows \underline{\mathcal{B}} \colon R$.

Proposition 2.2.4 (Exercise)**.** *Given an n-adjunction as above, for all objects $a \in \mathcal{A}$ and $b \in \mathcal{B}$, there is a natural isomorphism in $\mathcal{C}at_{n-1}$*

$$\mathrm{Hom}_{\mathcal{A}}(a, Rb) \cong \mathrm{Hom}_{\mathcal{B}}(La, b).$$

Example 2.2.5. Given a map $f\colon X \to Y$ in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$, we have an n-adjunction

$$f_!\colon \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/X} \rightleftarrows \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/Y} \colon f^*.$$

2.3. **Straightening-unstraightening.** We now turn to studying the n-category $[\mathcal{C}, \underline{\mathcal{C}at}_{n-1}]$ of n-copresheaves over an n-category $\mathcal{C}$. For this, we need a fibrational approach.

Definition 2.3.1. A map $p\colon A \to X$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is a **double** $(n-1)$**-left fibration** if, for every $m \geq 1$, the following commutative square is a pullback square in $\mathcal{C}at_{n-1}$.

$$\begin{tikzcd} A_m \arrow[r, "\langle 0\rangle^*"] \arrow[d, "p_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_0 \arrow[d, "p_0"] \\ X_m \arrow[r, "\langle 0\rangle^*"'] & X_0 \end{tikzcd}$$

We denote by $\mathcal{D}bl\mathcal{LF}ib(X)$ the full subcategory of the slice category $\mathcal{C}at_{n-1/X}^{\Delta^{\mathrm{op}}}$ spanned by the double $(n-1)$-left fibrations over X.

Similarly, we define **double** $(n-1)$**-right fibrations** (replacing $\langle 0\rangle^*$ by $\langle m\rangle^*$) and $\mathcal{D}bl\mathcal{RF}ib(X)$.

We focus on double $(n-1)$-left fibrations, but everything can be dualized to double $(n-1)$-right fibrations.

Remark 2.3.2. When $n = 1$, this retrieves the usual notion of left fibrations of categories (regarded as simplicial objects in groupoids).

Remark 2.3.3. The category $\mathcal{D}bl\mathcal{LF}ib(X)$ also defines an object of $\mathcal{C}at^{\otimes}$ with the same tensors, hom $(n-1)$-categories, and cotensors as those of $\mathcal{C}at_{n-1/X}^{\Delta^{\mathrm{op}}}$ (Exercise). So we have an associated n-category $\underline{\mathcal{D}bl\mathcal{LF}ib}(X)$ and the inclusion $I\colon \underline{\mathcal{D}bl\mathcal{LF}ib}(X) \hookrightarrow \underline{\mathcal{C}at}_{n-1/X}^{\Delta^{\mathrm{op}}}$ is fully faithful and preserves both tensors and cotensors.

Lemma 2.3.4. *Double $(n-1)$-left fibrations are closed under pullback and composition.*

Lemma 2.3.5. *Double $(n-1)$-left fibrations over the terminal n-category $[0]$ are precisely those of the form $\mathrm{cst}\,\mathcal{X} \to [0]$ for some $\mathcal{X} \in \mathcal{C}at_{n-1}$.*

Remark 2.3.6. Given a double $(n-1)$-left fibration $p\colon A \to X$ and an object $x \in X_0$, let $p^{-1}(x) \in \mathcal{C}at_{n-1}$ denote the fiber $\{x\} \times_{X_0} A_0$ of p_0 at x. We have a pullback square in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$

$$\begin{tikzcd} \mathrm{cst}(p^{-1}(x)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ {[0]} \arrow[r, "x"'] & X \end{tikzcd}$$

Hence the fibers of p are $(n-1)$-categories.

Over an n-category, a double $(n-1)$-left fibration is determined by its fibers.

Lemma 2.3.7. *Given an n-category $\mathcal{C}$, the following are equivalent for a map $A \to B$ between double $(n-1)$-left fibrations $p\colon A \to \mathcal{C}$ and $q\colon B \to \mathcal{C}$:*

(i) it is an isomorphism in $\mathcal{D}bl\mathcal{LF}ib(\mathcal{C})$;
(ii) the induced map $A_0 \to B_0$ is an isomorphism in $\mathcal{C}at_{n-1}$;
(iii) for every object $x \in \mathcal{C}$, the induced map $p^{-1}(x) \to q^{-1}(x)$ between fibers is an isomorphism in $\mathcal{C}at_{n-1}$.

Remark 2.3.8. We can assign to each double $(n-1)$-left fibration $p\colon A \to \mathcal{C}$ over an n-category $\mathcal{C}$ a functor $\mathcal{C} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$. It sends an object $x \in \mathcal{C}$ to the fiber $p^{-1}(x) \in \mathcal{C}at_{n-1}$. Given objects $x, y \in \mathcal{C}$, the composite in $\mathcal{C}at_{n-1}$

$$\mathcal{C}_1 \times_{\mathcal{C}_0} A_0 \xleftarrow{\cong} A_1 \xrightarrow{\langle 1\rangle^*} A_0$$

restricts to a map
$$\mathrm{Hom}_{\mathcal{C}}(x,y) \times p^{-1}(x) \to p^{-1}(y)$$
which, by cartesian closedness, corresponds to a map
$$\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}at_{n-1}}(p^{-1}(x), p^{-1}(y)).$$
This gives the action of the functor on hom $(n-1)$-categories. The higher pullback conditions allow us to encode all coherences for functoriality.

The process described above is called *straightening* and is in fact an equivalence. We introduce the corresponding *unstraightening* or *Grothendieck construction.*

Theorem 2.3.9. *Given an n-category $\mathcal{C}$, there is a fully faithful functor in $\mathcal{C}at_n$*
$$\textstyle\int_{\mathcal{C}} \colon [\mathcal{C}, \underline{\mathcal{C}at}_{n-1}] \to \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}$$
which restricts to an isomorphism in $\mathcal{C}at_n$
$$\textstyle\int_{\mathcal{C}} \colon [\mathcal{C}, \underline{\mathcal{C}at}_{n-1}] \xrightarrow{\cong} \underline{\mathcal{D}bl\mathcal{L}\mathcal{F}ib}(\mathcal{C}).$$
Given a functor $F \colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$, the following square in $\mathcal{C}at_n$ commutes
$$\begin{tikzcd}
{[\mathcal{D}, \underline{\mathcal{C}at}_{n-1}]} \arrow[r, "\int_{\mathcal{D}}"] \arrow[d, "F^*"'] & \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{D}} \arrow[d, "F^*"] \\
{[\mathcal{C}, \underline{\mathcal{C}at}_{n-1}]} \arrow[r, "\int_{\mathcal{C}}"'] & \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}
\end{tikzcd}$$

Remark 2.3.10. When $n = 1$ and $\mathcal{C}$ is a 1-category, this retrieves the usual straightening-unstraightening between functors $\mathcal{C} \to \mathcal{G}pd$ and left fibrations over $\mathcal{C}$.

Remark 2.3.11. Given a functor $F \colon \mathcal{C} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$, there is a pullback square in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$
$$\begin{tikzcd}
\int_{\mathcal{C}} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int_{\underline{\mathcal{C}at}_{n-1}} \mathrm{id} \arrow[d] \\
\mathcal{C} \arrow[r, "F"'] & \underline{\mathcal{C}at}_{n-1}
\end{tikzcd}$$
where $\mathrm{id} \colon \underline{\mathcal{C}at}_{n-1} \to \underline{\mathcal{C}at}_{n-1}$ is the identity functor (we will compute $\int_{\underline{\mathcal{C}at}_{n-1}} \mathrm{id}$ explicitely in Lecture 4). One can take this as the definition of the Grothendieck construction.

3. Lecture 3: Theory of fibrations and Quillen's Theorem A

3.1. **Double $(n-1)$-covariant equivalences.** We now turn to studying the notion of covariant equivalences associated to double $(n-1)$-left fibrations.

Proposition 3.1.1. *Given $X \in \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$, there is an adjunction*
$$L_X \colon \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X} \rightleftarrows \mathcal{D}bl\mathcal{L}\mathcal{F}ib(X) : I$$
where I denotes the full inclusion. Since I preserves cotensors, we also have an n-adjunction
$$L_X \colon \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/X} \rightleftarrows \underline{\mathcal{D}bl\mathcal{L}\mathcal{F}ib}(X) : I$$

Definition 3.1.2. Given $X \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, a map in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/X}$ is a **double $(n-1)$-covariant equivalence over** X if its image under the localization functor $L_X \colon \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/X} \to \mathcal{D}bl\mathcal{LF}ib(X)$ is an isomorphism in $\mathcal{D}bl\mathcal{LF}ib(X)$.

The dual notion for $\mathcal{D}bl\mathcal{RF}ib(X)$ is called a **double $(n-1)$-contravariant equivalence**.

As a consequence of the above n-adjunction, we have the following characterization.

Lemma 3.1.3. *Given $X \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, a map $f \colon A \to B$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/X}$ is a double $(n-1)$-covariant equivalence over X if and only if, for every double $(n-1)$-left fibration $p \colon C \to X$,*

$$f^* \colon \operatorname{Hom}_{\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/X}}(B, C) \to \operatorname{Hom}_{\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/X}}(A, C)$$

is an isomorphism in $\mathcal{C}at_{n-1}$.

Lemma 3.1.4. *Double $(n-1)$-covariant equivalences are closed under colimits and 2-out-of-3.*

Lemma 3.1.5. *Given a map $f \colon X \to Y$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, the postcomposition functor*

$$f_! \colon \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/X} \to \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/Y}$$

preserves double $(n-1)$-covariant equivalences.

Proof. The below left diagram of right adjoint functors commutes, and hence so does the below right diagram of corresponding left adjoint functors.

$$\begin{tikzcd} \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/X} & \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/Y} \arrow[l, "f^*"'] \\ \mathcal{D}bl\mathcal{LF}ib(X) \arrow[u, hook] & \mathcal{D}bl\mathcal{LF}ib(Y) \arrow[l, "f^*"] \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/X} \arrow[r, "f_!"] \arrow[d, "L_X^{\mathcal{L}}"'] & \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}{}_{/Y} \arrow[d, "L_Y^{\mathcal{L}}"] \\ \mathcal{D}bl\mathcal{LF}ib(X) \arrow[r, "L_Y^{\mathcal{L}} f_!"'] & \mathcal{D}bl\mathcal{LF}ib(Y) \end{tikzcd}$$

□

3.2. **Double $(n-1)$-initial objects.** We now turn to defining double $(n-1)$-initial objects, which will provide our main examples of double $(n-1)$-covariant equivalences.

Notation 3.2.1. We write $\mathcal{D}bl\mathcal{C}at_{n-1} \coloneqq \mathcal{S}eg(\mathcal{C}at_{n-1})$ for the category of Segal objects in $\mathcal{C}at_{n-1}$, which we refer to as **double $(n-1)$-categories**.

Remark 3.2.2. If X is a double $(n-1)$-category and $p \colon A \to X$ is a double $(n-1)$-left fibration, then A is also a double $(n-1)$-category. Note that, if X is an n-category, then A is a double $(n-1)$-category, but almost never an n-category as A_0 is almost never a groupoid.

Definition 3.2.3. Given $m, m' \geq 0$, the **join** of $[m]$ and $[m']$ is the object of Δ

$$[m] \star [m'] \coloneqq [m + 1 + m'].$$

It comes with canonical inclusions $[m] \to [m] \star [m']$ and $[m'] \to [m] \star [m']$ given by

$$\langle 0, \dots, m \rangle \colon [m] \to [m + 1 + m'] \quad \text{and} \quad \langle m+1, \dots, m+1+m' \rangle \colon [m'] \to [m+1+m'].$$

This extends to a functor $- \star - \colon \Delta \times \Delta \to \Delta$ and natural transformations $\pi_i \Rightarrow - \star -$ for $i = 0, 1$.

Definition 3.2.4. Given $X \in \mathcal{D}bl\mathcal{C}at_{n-1}$ and an object $x \in X_0$, we define the **slice double $(n-1)$-category under** x to be the simplicial object

$$^{x/}X \colon \Delta^{\mathrm{op}} \to \mathcal{C}at_{n-1}, \quad [m] \mapsto \operatorname{Hom}_{[0]/\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}}([0] \star [m], X)$$

where the join comes with the canonical inclusion $[0] \to [0] \star [m]$ and X with the map $x \colon [0] \to X$.

The canonical inclusions $[m] \to [0] \star [m]$ induce a natural projection $^{x/}X \to X$, which is a double $(n-1)$-left fibration.

Definition 3.2.5. Given an object $X \in \mathcal{D}bl\mathcal{C}at_{n-1}$, an object $x \in X_0$ is **double** $(n-1)$**-initial** if the canonical projection ${}^{x/}X \to X$ is an isomorphism in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$.

Remark 3.2.6. Given an n-category $\mathcal{C}$, an object $x \in \mathcal{C}$ is double $(n-1)$-initial if and only if, for every object $y \in \mathcal{C}$, there is an isomorphism in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \cong [0].$$

Remark 3.2.7. When $n = 1$, this retrieves the usual notions of slice left fibrations (or slices under an object) and initial objects in categories (regarded as simplicial objects in groupoids).

We now want to prove that an object $x \in X_0$ is double $(n-1)$-initial if and only if the corresponding map $x\colon [0] \to X$ is a double $(n-1)$-covariant equivalence over X.

Definition 3.2.8. A map $f\colon A \to B$ in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ is a **covariant strong deformation retract** if there is a retraction $r\colon B \to A$ such that $rf = \mathrm{id}_A$ and a map $h\colon B \times [1] \to B$ that fits in the following commutative diagram in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$.

$$\begin{tikzcd}
B \arrow[r, "\mathrm{id}_B \times \langle 0 \rangle"] \arrow[d, "r"'] & B \times [1] \arrow[d, "h"'] & B \arrow[l, "\mathrm{id}_B \times \langle 1 \rangle"'] \arrow[dl, "\mathrm{id}_B"] \\
A \arrow[r, "f"'] & B &
\end{tikzcd}$$

Proposition 3.2.9 (Exercise)**.** *If $f\colon A \to B$ is a covariant strong deformation retract in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ and $B \to X$ is a map in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, then f is a double $(n-1)$-covariant equivalence over X.*

Lemma 3.2.10. *Given $X \in \mathcal{D}bl\mathcal{C}at_{n-1}$, if an object $x \in X_0$ is double $(n-1)$-initial, then the induced map $x\colon [0] \to X$ is a covariant strong deformation retract in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$.*

Proof. First note that the map $x\colon [0] \to X$ admits a retraction given by the unique map $X \to [0]$. Next, observe that there is a commutative square in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ (given by the solid arrows)

$$\begin{tikzcd}
{}^{x/}X \arrow[r] \arrow[d] & X^{[1]} \arrow[d, "(\langle 0 \rangle^*, \langle 1 \rangle^*)"] \\
X \arrow[r, "x \times \mathrm{id}_X"'] \arrow[ur, dashed, "h"] & X \times X
\end{tikzcd}$$

where the top map is induced by the maps $[1] \times [m] \to [0] \star [m] = [m+1]$ sending $(0, j) \mapsto 0$ and $(1, j) \mapsto j+1$ for all $0 \leq j \leq m$. Since $x \in X_0$ is double $(n-1)$-initial, the canonical projection ${}^{x/}X \to X$ is an isomorphism in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ and so we have a lift h in the above commutative square. By cartesian closedness of $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, the lift h induces a map fitting in the commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X \times \langle 0 \rangle"] \arrow[d] & X \times [1] \arrow[d, "h"'] & X \arrow[l, "\mathrm{id}_X \times \langle 1 \rangle"'] \arrow[dl, "\mathrm{id}_X"] \\
{[0]} \arrow[r, "x"'] & X &
\end{tikzcd}$$

□

Proposition 3.2.11. *Given $X \in \mathcal{D}bl\mathcal{C}at_{n-1}$ and an object $x \in X_0$, the map*

$$\begin{tikzcd}
{[0]} \arrow[rr, "\mathrm{id}_x"] \arrow[dr, "x"'] & & {}^{x/}X \arrow[dl] \\
& X &
\end{tikzcd}$$

is a double $(n-1)$-covariant equivalence over X.

Proof. This follows from the previous lemma, using that $\mathrm{id}_x \in {}^{x/}X$ is double $(n-1)$-initial and that covariant strong deformation retracts are double $(n-1)$-covariant equivalences. $\square$

Proposition 3.2.12. *Given an object $X \in \mathcal{D}bl\mathcal{C}at_{n-1}$, an object $x \in X_0$ is double $(n-1)$-initial if and only if the induced map $x\colon [0] \to X$ is a double $(n-1)$-covariant equivalence over X.*

Proof. This follows by the 2-out-of-3 property for double $(n-1)$-covariant equivalences applied to the commutative triangle in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X}$

$$\begin{tikzcd} {[0]} \arrow[rr, "\mathrm{id}_x"] \arrow[dr, "x"'] & & {}^{x/}X \arrow[dl] \\ & X & \end{tikzcd}$$

$\square$

3.3. **Quillen's Theorem A.** Finally, we turn to proving a version of Quillen's Theorem A which provides a useful characterization of double $(n-1)$-covariant equivalences. We first state the following result without proof.

Proposition 3.3.1. *Given a double $(n-1)$-left fibration $p\colon X \to \mathcal{C}$ over an n-category $\mathcal{C}$, the pullback functor*

$$p^*\colon \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}} \to \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X}$$

preserves double $(n-1)$-contravariant equivalences.

Remark 3.3.2. Similarly, the pullback functor along a double $(n-1)$-right fibration over an n-category preserves double $(n-1)$-covariant equivalences.

Lemma 3.3.3. *Given $X \in \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$, the functor*

$$\mathrm{colim}_{\Delta^{\mathrm{op}}}\colon \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X} \to \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1} \xrightarrow{\mathrm{colim}_{\Delta^{\mathrm{op}}}} \mathcal{C}at_{n-1}$$

sends double $(n-1)$-covariant/contravariant equivalences to isomorphisms.

Proof. We prove the result in the covariant case, and the contravariant case works similarly. Recall that $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X} \to \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1} \xrightarrow{\mathrm{colim}_{\Delta^{\mathrm{op}}}} \mathcal{C}at_{n-1}$ is left adjoint to the functor

$$X \otimes (-)\colon \mathcal{C}at_{n-1} \xrightarrow{\mathrm{cst}} \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1} \xrightarrow{X \times (-)} \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X}$$

which takes values in the full subcategory $\mathcal{D}bl\mathcal{LF}ib(X) \subseteq \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X}$. Since the below left diagram of right adjoints commutes, then so does the below right diagram of corresponding left adjoints.

$$\begin{tikzcd} \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X} & \\ & \mathcal{C}at_{n-1} \arrow[ul, "X \otimes (-)"'] \arrow[dl, "X \otimes (-)"] \\ \mathcal{D}bl\mathcal{LF}ib(X) \arrow[uu, hook] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/X} \arrow[dr, "\mathrm{colim}_{\Delta^{\mathrm{op}}}"] \arrow[dd, "L^{\mathcal{L}}_X"'] & \\ & \mathcal{C}at_{n-1} \\ \mathcal{D}bl\mathcal{LF}ib(X) \arrow[ur, "\mathrm{colim}_{\Delta^{\mathrm{op}}}"'] & \end{tikzcd}$$

$\square$

Lemma 3.3.4. *Given an n-category $\mathcal{C}$, an object $x \in \mathcal{C}$, and a map $p\colon A \to \mathcal{C}$ in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$, there is a natural isomorphism in $\mathcal{C}at_{n-1}$*

$$\mathrm{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} A) \cong \{x\} \times_{\mathcal{C}_0} (L_{\mathcal{C}}A)_0$$

Proof. Consider the maps in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$

$$\mathcal{C}_{/x} \times_{\mathcal{C}} A \xrightarrow{\mathcal{C}_{/x} \times_{\mathcal{C}} \eta_p} \mathcal{C}_{/x} \times_{\mathcal{C}} L_{\mathcal{C}}A \xleftarrow{\mathrm{id}_x \times_{\mathcal{C}} L_{\mathcal{C}}A} \{x\} \times_{\mathcal{C}} L_{\mathcal{C}}A$$

where $\eta_p\colon A \to L_{\mathcal{C}}A$ is the component of the unit of the adjunction $L_{\mathcal{C}} \dashv I$ at $p\colon A \to \mathcal{C}$. Since $\mathcal{C}_{/x} \to \mathcal{C}$ is a double $(n-1)$-right fibration, the pullback $\mathcal{C}_{/x} \times_{\mathcal{C}} \eta_p$ of the double $(n-1)$-covariant equivalence $\eta_p\colon A \to L_{\mathcal{C}}A$ over $\mathcal{C}$ is a double $(n-1)$-covariant equivalence over $\mathcal{C}_{/x}$. Since $L_{\mathcal{C}}A \to \mathcal{C}$ is a double $(n-1)$-left fibration, the pullback $\mathrm{id}_x \times_{\mathcal{C}} L_{\mathcal{C}}A$ of the double $(n-1)$-contravariant equivalence $\mathrm{id}_x\colon [0] \to \mathcal{C}_{/x}$ over $\mathcal{C}$ is a contravariant equivalence over $L_{\mathcal{C}}A$. Hence, by the previous lemma, we get natural isomorphisms in $\mathcal{C}at_{n-1}$

$$\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} A) \xrightarrow[\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} \eta_p)]{\cong} \operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} L_{\mathcal{C}}A) \xleftarrow[\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathrm{id}_x \times_{\mathcal{C}} L_{\mathcal{C}}A)]{\cong} \operatorname{colim}_{\Delta^{\mathrm{op}}}(\{x\} \times_{\mathcal{C}} L_{\mathcal{C}}A)$$

Finally, since $\{x\} \times_{\mathcal{C}} L_{\mathcal{C}}A \cong \mathrm{cst}(\{x\} \times_{\mathcal{C}_0} (L_{\mathcal{C}}A)_0)$, we deduce that

$$\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} A) \cong \operatorname{colim}_{\Delta^{\mathrm{op}}}(\{x\} \times_{\mathcal{C}} L_{\mathcal{C}}A) \cong \{x\} \times_{\mathcal{C}_0} (L_{\mathcal{C}}A)_0. \qquad \square$$

Theorem 3.3.5 (Quillen's Theorem A)**.** *Given an n-category $\mathcal{C}$, a map $f\colon A \to B$ in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}$ is a double $(n-1)$-covariant equivalence over $\mathcal{C}$ if and only if, for all $x \in \mathcal{C}$, the induced map*

$$\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} A) \to \operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} B)$$

is an isomorphism in $\mathcal{C}at_{n-1}$.

Proof. By definition, a map $f\colon A \to B$ in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}$ is a double $(n-1)$-covariant equivalence over $\mathcal{C}$ if and only if the induced map $L_{\mathcal{C}}f\colon L_{\mathcal{C}}A \to L_{\mathcal{C}}B$ is an isomorphism in $\mathcal{D}bl\mathcal{LF}ib(\mathcal{C})$. This holds if and only if the top map in the commutative square

$$\begin{tikzcd}
\{x\} \times_{\mathcal{C}_0} (L_{\mathcal{C}}A)_0 \arrow[r, "\{x\} \times_{\mathcal{C}_0} (L_{\mathcal{C}}f)_0"] & \{x\} \times_{\mathcal{C}_0} (L_{\mathcal{C}}B)_0 \\
\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} A) \arrow[u, "\cong"] \arrow[r, "\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} f)"'] & \operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathcal{C}_{/x} \times_{\mathcal{C}} B) \arrow[u, "\cong"']
\end{tikzcd}$$

is an isomorphism in $\mathcal{C}at_{n-1}$, where the vertical morphisms are isomorphisms by the previous lemma. By 2-out-of-3, this holds if and only if the bottom map in the above diagram is an isomorphism in $\mathcal{C}at_{n-1}$. $\square$

4. Lecture 4: Yoneda Lemma and weighted (co)limits

4.1. **Hom functor.** We now construct the hom-functor $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \underline{\mathcal{C}at}_{n-1}$ of an n-category $\mathcal{C}$. Let us first define the opposite of an n-category.

Definition 4.1.1. There is an involution $(-)^{\mathrm{op}}\colon \Delta \to \Delta$ which is the identity on objects and sends a map $\alpha\colon [m] \to [n]$ in Δ to the map $\langle n-\alpha(m), \ldots, n-\alpha(0)\rangle\colon [m] \to [n]$ in Δ. It induces by precomposition an involution

$$(-)^{\mathrm{op}}\colon \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1} \to \mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$$

which restricts to an involution $(-)^{\mathrm{op}}\colon \mathcal{C}at_n \to \mathcal{C}at_n$.

Given an n-category $\mathcal{C}$, its **opposite n-category** $\mathcal{C}^{\mathrm{op}}$ has

- ▷ the same underlying groupoid as $\mathcal{C}$, namely $\mathcal{C}_0$;
- ▷ hom $(n-1)$-categories $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x,y) \cong \mathrm{Hom}_{\mathcal{C}}(y,x)$ for all objects $x, y \in \mathcal{C}$.

Remark 4.1.2. Given an n-category $\mathcal{C}$, we have that $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \cong \underline{\mathcal{D}bl\mathcal{LF}ib}(\mathcal{C}^{\mathrm{op}}) \cong \underline{\mathcal{D}bl\mathcal{RF}ib}(\mathcal{C})$.

Definition 4.1.3. Given an n-category $\mathcal{C}$, we define the **twisted arrow double $(n-1)$-category of** $\mathcal{C}$ to be the simplicial object

$$\mathbb{T}\mathrm{w}(\mathcal{C})\colon \Delta^{\mathrm{op}} \to \mathcal{C}at_{n-1}, \quad [m] \mapsto \mathrm{Hom}_{\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}}([m]^{\mathrm{op}} \star [m], \mathcal{C}).$$

The canonical inclusions $[m]^{\mathrm{op}} \amalg [m] \to [m]^{\mathrm{op}} \star [m]$ induce a natural projection $\mathbb{T}\mathrm{w}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, which is a double $(n-1)$-left fibration.

This extends to a functor $\mathbb{T}\mathrm{w}\colon \mathcal{C}at_n \to \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$ and a natural transformation $\mathbb{T}\mathrm{w} \Rightarrow (-)^{\mathrm{op}} \times (-)$.

Remark 4.1.4. The fiber of $\mathbb{T}\mathrm{w}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ at an object $(x, y) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is given by the hom $(n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(x, y)$ since the map $\mathbb{T}\mathrm{w}(\mathcal{C})_0 \to (\mathcal{C}^{\mathrm{op}})_0 \times \mathcal{C}_0$ in $\mathcal{C}at_{n-1}$ can be identified with the map $(\langle 0\rangle^*, \langle 1\rangle^*)\colon \mathcal{C}_1 \to \mathcal{C}_0 \times \mathcal{C}_0$.

Construction 4.1.5. Given an n-category $\mathcal{C}$, by straightening the double $(n-1)$-left fibration $\mathbb{T}\mathrm{w}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, we obtain the **hom-functor of** $\mathcal{C}$ in $\mathcal{C}at_n$

$$\mathrm{Hom}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \underline{\mathcal{C}at}_{n-1},$$

which is given on objects by sending $(x, y) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(x, y)$.

Proposition 4.1.6. *Given an n-category $\mathcal{C}$ and an object $x \in \mathcal{C}$, there is a pullback square in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$*

$$\begin{tikzcd} {}^{x/}\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[d] \\ \mathcal{C} \arrow[r, "\{x\} \times \mathrm{id}_{\mathcal{C}}"'] & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \end{tikzcd}$$

In particular, we have an isomorphism $\int_{\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(x, -) \cong {}^{x/}\mathcal{C}$ in $\mathcal{C}at_{n-1/\mathcal{C}}^{\Delta^{\mathrm{op}}}$

Lemma 4.1.7. *The unstraightening of the identity functor* $\mathrm{id}\colon \underline{\mathcal{C}at}_{n-1} \to \underline{\mathcal{C}at}_{n-1}$ *in $\mathcal{C}at_n$ is the slice double $(n-1)$-left fibration ${}^{[0]/}\underline{\mathcal{C}at}_{n-1} \to \underline{\mathcal{C}at}_{n-1}$.*

Proof. Since $[0]$ is the unit of the cartesian monoidal structure on $\mathcal{C}at_{n-1}$, the functor

$$\mathrm{Hom}_{\mathcal{C}at_{n-1}}([0], -)\colon \underline{\mathcal{C}at}_{n-1} \to \underline{\mathcal{C}at}_{n-1}$$

is canonically identified with the identity functor. □

Corollary 4.1.8. *Given functors $F\colon \mathcal{C} \to \underline{\mathcal{C}at}_{n-1}$, there is a pullback square in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$*

$$\begin{tikzcd} \int_{\mathcal{C}} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {}^{[0]/}\underline{\mathcal{C}at}_{n-1} \arrow[d] \\ \mathcal{C} \arrow[r, "F"'] & \underline{\mathcal{C}at}_{n-1} \end{tikzcd}$$

4.2. **Yoneda lemma.** We now turn to proving the n-categorical Yoneda Lemma; namely that, for an n-category $\mathcal{C}$, there is an isomorphism in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(\mathrm{Hom}_{\mathcal{C}}(-, x), F) \cong Fx$$

which is appropriately natural in $x \in \mathcal{C}$ and $F\colon \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$.

Remark 4.2.1. Given an n-category $\mathcal{C}$, we refer to the 1-morphisms of the n-category $[\mathcal{C}, \underline{\mathcal{C}at}_{n-1}]$ as **n-natural transformations**. Since the functor $\int_{\mathcal{C}}\colon [\mathcal{C}, \underline{\mathcal{C}at}_{n-1}] \to \underline{\mathcal{C}at}_{n-1/\mathcal{C}}^{\Delta^{\mathrm{op}}}$ is fully faithful, the data of an n-natural transformation $F \Rightarrow G$ of functors $F, G\colon \mathcal{C} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$ is the same as the data of a map $\int_{\mathcal{C}} F \to \int_{\mathcal{C}} G$ in $\mathcal{C}at_{n-1/\mathcal{C}}^{\Delta^{\mathrm{op}}}$.

By the characterization of isomorphisms in $\mathcal{D}bl\mathcal{LF}ib(\mathcal{C})$, the invertible n-natural transformations in $[\mathcal{C}, \underline{\mathcal{C}at}_{n-1}]$ are precisely the ones that are pointwise isomorphisms in $\mathcal{C}at_{n-1}$. In this case, we write either "an **n-natural isomorphism** $F \cong G$ of functors $\mathcal{C} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$" or "for all objects $x \in \mathcal{C}$, an isomorphism $Fx \cong Gx$ in $\mathcal{D}$ which is n-natural in $x \in \mathcal{C}$".

Construction 4.2.2. Given an n-category $\mathcal{C}$, the hom-functor $\mathrm{Hom}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \underline{\mathcal{C}at}_{n-1}$ induces by cartesian closedness of $\mathcal{C}at_n$ the **n-Yoneda functor** in $\mathcal{C}at_n$

$$y_{\mathcal{C}}\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$$

which is given on objects by sending $x \in \mathcal{C}$ to the representable $\mathrm{Hom}_{\mathcal{C}}(-,x)\colon \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$.

Proposition 4.2.3 (Pointwise n-Yoneda lemma)**.** *Given an n-category $\mathcal{C}$, an object $x \in \mathcal{C}$, there is an n-natural isomorphism of functors $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$*

$$\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(\mathrm{Hom}_{\mathcal{C}}(-,x), -) \cong \mathrm{ev}_x,$$

or equivalently, for all functors $F\colon \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$, there is an isomorphism in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(\mathrm{Hom}_{\mathcal{C}}(-,x), F) \cong Fx = \mathrm{ev}_x(F)$$

which is n-natural in $F \in [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$.

Proof. Consider the composite in $\mathcal{C}at_n$

$$(\underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}})^{\mathrm{op}} \xrightarrow{y_{(\underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}})^{\mathrm{op}}}} [\underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}, \underline{\mathcal{C}at}_{n-1}] \xrightarrow{I^*} [\underline{\mathcal{D}bl\mathcal{RF}ib}(\mathcal{C}), \underline{\mathcal{C}at}_{n-1}]$$

of the n-Yoneda functor and the functor obtained by precomposition with the canonical inclusion $I\colon \underline{\mathcal{D}bl\mathcal{RF}ib}(\mathcal{C}) \hookrightarrow \underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}$. It sends the contravariant equivalence $\mathrm{id}_x\colon [0] \to \mathcal{C}_{/x}$ in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}$ the n-natural transformation of functors $\underline{\mathcal{D}bl\mathcal{RF}ib}(\mathcal{C}) \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$

$$y_{(\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}})^{\mathrm{op}}}(\mathcal{C}_{/x}) = \mathrm{Hom}_{\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}}(\mathcal{C}_{/x}, -) \Rightarrow \mathrm{Hom}_{\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}}([0], -) = y_{(\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}})^{\mathrm{op}}}([0]).$$

By the characterization of contravariant equivalences, this is pointwise an isomorphism in $\mathcal{C}at_{n-1}$, i.e., for every double $(n-1)$-left fibration $p\colon A \to \mathcal{C}$, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}}(\mathcal{C}_{/x}, A) \cong \mathrm{Hom}_{\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}}([0], A)$$

which is n-natural in $p \in \underline{\mathcal{D}bl\mathcal{RF}ib}(\mathcal{C})$. Using the isomorphism $\int_{\mathcal{C}}\colon [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \xrightarrow{\cong} \underline{\mathcal{D}bl\mathcal{RF}ib}(\mathcal{C})$ in $\mathcal{C}at_n$, we get, for every functor $F\colon \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$, an isomorphism in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}}(\mathcal{C}_{/x}, \textstyle\int_{\mathcal{C}} F) \cong \mathrm{Hom}_{\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}}([0], \textstyle\int_{\mathcal{C}} F)$$

which is n-natural in $F \in [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$. Using the computations $\mathcal{C}_{/x} \cong \int_{\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(-,x)$ and $\mathrm{Hom}_{\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/\mathcal{C}}}([0], \int_{\mathcal{C}} F) \cong Fx$, and that $\int_{\mathcal{C}}$ is fully faithful, the above can be identified with, for every functor $F\colon \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$, an isomorphism in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(\mathrm{Hom}_{\mathcal{C}}(-,x), F) \cong Fx,$$

which is n-natural in $F \in [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$. ☐

To prove that it is further natural in $x \in \mathcal{C}$, we will use the following key lemma twice.

Lemma 4.2.4 (Exercise)**.** *Given an n-category $\mathcal{C}$ and a functor $F\colon \mathcal{D} \to \mathcal{E}$ in $\mathcal{C}at_n$, consider a commutative square in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$ as below left, where p and q are double $(n-1)$-left fibrations.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ \mathcal{C} \times \mathcal{D} \arrow[r, "\mathrm{id}_{\mathcal{C}} \times F"'] & \mathcal{C} \times \mathcal{E} \end{tikzcd} \qquad \begin{tikzcd} \{x\} \times_{\mathcal{C}} A \arrow[r, "\{x\} \times_{\mathcal{C}} f"] \arrow[d, "\{x\} \times_{\mathcal{C}} p"'] & \{x\} \times_{\mathcal{C}} B \arrow[d, "\{x\} \times_{\mathcal{C}} q"] \\ \mathcal{D} \arrow[r, "F"'] & \mathcal{E} \end{tikzcd}$$

Then f is a double $(n-1)$-covariant equivalence over $\mathcal{C} \times \mathcal{E}$ if and only if, for every object $x \in \mathcal{C}$, the induced map $\{x\} \times_{\mathcal{C}} f$ (as above right) is a double $(n-1)$-covariant equivalence over $\mathcal{E}$.

Lemma 4.2.5 (Exercise)**.** *Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$, define u_F to be the unique map given by the universal property of pullbacks fitting in the commutative diagram in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$.*

$$\begin{tikzcd} \mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[r, dashed, "u_F"'] \arrow[rr, bend left, "\mathbb{T}\mathrm{w}(F)"] \arrow[d] & \int_{\mathcal{C}^{\mathrm{op}} \times \mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(F(-), -) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{T}\mathrm{w}(\mathcal{D}) \arrow[d] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \times F"'] & \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "F^{\mathrm{op}} \times \mathrm{id}_{\mathcal{D}}"'] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \end{tikzcd}$$

Then the map u_F is a double $(n-1)$-covariant equivalence over $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$.

As a special case, we get:

Corollary 4.2.6. *Given an n-category $\mathcal{C}$, in the commutative diagram in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$*

$$\begin{tikzcd} \mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[r, "u_{y_{\mathcal{C}}}"] \arrow[d] & \int_{\mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]} \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(y_{\mathcal{C}}(-), -) \arrow[d] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \times y_{\mathcal{C}}"'] & \mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \end{tikzcd}$$

the map $u_{y_{\mathcal{C}}}$ is a double $(n-1)$-covariant equivalence over $\mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$, where $y_{\mathcal{C}}\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ is the n-Yoneda functor.

Lemma 4.2.7. *Given an n-category $\mathcal{C}$, in the commutative diagram in $\mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$*

$$\begin{tikzcd} \mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[r, dashed, "v"] \arrow[rr, bend left] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \int_{\mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]} \mathrm{ev} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {}^{[0]/}\mathcal{C}at_{n-1} \arrow[d] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \times y_{\mathcal{C}}"'] \arrow[rr, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,-)"'] & \mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \arrow[r, "\mathrm{ev}"'] & \mathcal{C}at_{n-1} \end{tikzcd}$$

the map v is a double $(n-1)$-covariant equivalence over $\mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$.

Proof. By Lemma 4.2.4, it is sufficient to show that, for every object $x \in \mathcal{C}$, the induced map

$$\begin{tikzcd} {}^{x/}\mathcal{C} \cong \{x\} \times_{\mathcal{C}^{\mathrm{op}}} \mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[r, "\{x\} \times_{\mathcal{C}^{\mathrm{op}}} v"] \arrow[d] & \{x\} \times_{\mathcal{C}^{\mathrm{op}}} \int_{\mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]} \mathrm{ev} \cong \int_{[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]} \mathrm{ev}_x \arrow[d] \\ \mathcal{C} \arrow[r, "y_{\mathcal{C}}"'] & [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \end{tikzcd}$$

is a double $(n-1)$-covariant equivalence over $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$. By the pointwise n-Yoneda lemma, we have isomorphisms in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1/[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]}$

$$\textstyle\int_{[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]} \mathrm{ev}_x \cong \int_{[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]} \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]}(\mathrm{Hom}_{\mathcal{C}}(-,x),-) \cong {}^{\mathrm{Hom}_{\mathcal{C}}(-,x)/}[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$$

As the map $\{x\} \times_{\mathcal{C}^{\mathrm{op}}} v$ sends the object $(x, \mathrm{id}_x) \in {}^{x/}\mathcal{C}$ to $(\mathrm{Hom}_{\mathcal{C}}(-,x), \mathrm{id}_x) \in \int_{[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]} \mathrm{ev}_x$, it can be identified with the map $y_{\mathcal{C}} \colon {}^{x/}\mathcal{C} \to {}^{\mathrm{Hom}_{\mathcal{C}}(-,x)/}[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$. In the commutative diagram in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$,

$$\begin{tikzcd} & {[0]} \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-,x)}"] & \\ {}^{x/}\mathcal{C} \arrow[rr, "y_{\mathcal{C}}"'] & & {}^{\mathrm{Hom}_{\mathcal{C}}(-,x)/}[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \end{tikzcd}$$

both maps from $[0]$ are double $(n-1)$-covariant equivalences over $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$, and hence by 2-out-of-3 so is the map $y_{\mathcal{C}} \colon {}^{x/}\mathcal{C} \to {}^{\mathrm{Hom}_{\mathcal{C}}(-,x)/}[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$. □

Theorem 4.2.8. *Given an n-category $\mathcal{C}$, there is an n-natural isomorphism of functors $\mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$*

$$\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]}(y_{\mathcal{C}}(-),-) \cong \mathrm{ev},$$

or equivalently, for all objects $x \in \mathcal{C}$ and all functors $F \colon \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$, there is an isomorphism in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]}(\mathrm{Hom}_{\mathcal{C}}(-,x), F) \cong Fx = \mathrm{ev}_x(F)$$

which is n-natural in both $x \in \mathcal{C}^{\mathrm{op}}$ and $F \in [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$.

Proof. We have the following zig-zag of double $(n-1)$-covariant equivalences over $\mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$.

$$\begin{tikzcd} \int_{\mathcal{C}^{\mathrm{op}}\times[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]} \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]}(y_{\mathcal{C}}(-),-) \arrow[dr] & \mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[l, "u_{y_{\mathcal{C}}}"'] \arrow[r, "v"] \arrow[d] & \int_{\mathcal{C}^{\mathrm{op}}\times[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]} \mathrm{ev} \arrow[dl] \\ & \mathcal{C}^{\mathrm{op}} \times [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] & \end{tikzcd}$$

Since the left-hand and right-hand vertical maps are double $(n-1)$-left fibrations, their sources are in fact isomorphic, giving the desired n-natural isomorphism. □

Lemma 4.2.9 (Exercise)**.** *A functor $F \colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$ is fully faithful if and only if the following square is a pullback square in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$.*

$$\begin{tikzcd} \mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[r, "\mathbb{T}\mathrm{w}(F)"] \arrow[d] & \mathbb{T}\mathrm{w}(\mathcal{D}) \arrow[d] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "F^{\mathrm{op}} \times F"'] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \end{tikzcd}$$

Corollary 4.2.10. *The n-Yoneda functor*

$$y_{\mathcal{C}} \colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$$

*is fully faithful. We refer to it as the n-**Yoneda embedding**.*

Proof. Combining the above results, we see that the following are pullback squares in $\mathcal{C}at^{\Delta^{\mathrm{op}}}_{n-1}$.

$$\begin{tikzcd}
\mathbb{T}\mathrm{w}(\mathcal{C}) \arrow[r, "v"] \arrow[d] \arrow[rr, bend left, "\mathbb{T}\mathrm{w}(y_{\mathcal{C}})"] \arrow[dr, phantom, "\lrcorner", very near start] & \int_{\mathcal{C}^{\mathrm{op}}\times[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]} \mathrm{ev} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{T}\mathrm{w}([\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]) \arrow[d] \\
\mathcal{C}^{\mathrm{op}}\times\mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}^{\mathrm{op}}}\times y_{\mathcal{C}}"'] & \mathcal{C}^{\mathrm{op}}\times[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}] \arrow[r, "y_{\mathcal{C}}^{\mathrm{op}}\times\mathrm{id}"'] & {[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]^{\mathrm{op}}\times[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]}
\end{tikzcd}$$

□

Corollary 4.2.11. *Given n-categories $\mathcal{J}$ and $\mathcal{C}$, postcomposition with the n-Yoneda embedding*

$$(y_{\mathcal{C}})_*\colon [\mathcal{J},\mathcal{C}] \to [\mathcal{J},[\mathcal{C}^{\mathrm{op}},\mathcal{C}at_{n-1}]]$$

is fully faithful.

4.3. **Weighted (co)limits.** In n-category theory, the notion of *conical* (co)limit is insufficient, and we need to look at (co)limits whose cones are of a more general shape, encoded by a weight.

Definition 4.3.1. Let $F\colon \mathcal{J}\to\mathcal{C}$ and $W\colon\mathcal{J}\to\underline{\mathcal{C}at}_{n-1}$ be functors in $\mathcal{C}at_n$. A **W-weighted n-limit of** F is an object $\lim^W_{\mathcal{J}}F$ in $\mathcal{C}$ such that, for all objects $x\in\mathcal{C}$, there is an isomorphism in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{\mathcal{C}}(x,\lim^W_{\mathcal{J}}F)\cong\mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(W,\mathrm{Hom}_{\mathcal{C}}(x,F-))$$

which is n-natural in $x\in\mathcal{C}^{\mathrm{op}}$. The right-hand side is the functor in $\mathcal{C}at_n$ given by the composite

$$\mathcal{C}^{\mathrm{op}}\xrightarrow{y_{\mathcal{C}^{\mathrm{op}}}}[\mathcal{C},\underline{\mathcal{C}at}_{n-1}]\xrightarrow{F^*}[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]\xrightarrow{\mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(W,-)}\underline{\mathcal{C}at}_{n-1}.$$

Example 4.3.2. We have the following examples of weighted n-limits.

- ▷ **Conical n-limits:** By taking $W=\mathrm{cst}[0]\colon\mathcal{J}\to\underline{\mathcal{C}at}_{n-1}$ to be the constant functor at the terminal object, the notion of W-weighted n-limit corresponds to that of *conical* n-limit, denoted simply by $\lim_{\mathcal{J}}F$. Although this is not straightforward to see, it satisfies the following universal property: for all objects $x\in\mathcal{C}$, there is an isomorphism in $\mathcal{C}at_{n-1}$

  $$\mathrm{Hom}_{\mathcal{C}}(x,\lim_{\mathcal{J}}F)\cong\mathrm{Hom}_{[\mathcal{J},\mathcal{C}]}(\mathrm{cst}\,x,F)$$

  which is n-natural in $x\in\mathcal{C}^{\mathrm{op}}$. When $n=1$, this recovers the usual notion of limit.
- ▷ **Directed pullbacks/comma objects:** Let $n=2$, $\mathcal{J}=\{\bullet\to\bullet\leftarrow\bullet\}$ be the free category on a span and $W\colon\mathcal{J}\to\underline{\mathcal{C}at}_1$ be the diagram in $\mathcal{C}at$ given by

  $$[0]\xrightarrow{0}[1]\xleftarrow{1}[0].$$

  The W-weighted limit of a functor $F\colon\mathcal{J}\to\mathcal{C}$ in $\mathcal{C}at_2$, corresponding to a diagram in $\mathcal{C}$

  $$x\xrightarrow{f}y\xleftarrow{g}z$$

  is the directed pullback in $\mathcal{C}$:

  $$\begin{tikzcd}
  \lim^W_{\mathcal{J}}F \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & x \arrow[d, "f"] \\
  y \arrow[r, "g"'] & z
  \end{tikzcd}$$

  When $\mathcal{C}=\underline{\mathcal{C}at}_1$, this recovers the comma category construction $f\downarrow g$.
- ▷ **Lax limits:** Let $n=2$, $\mathcal{J}$ be a 1-category, and $W=\mathcal{J}_{/(-)}\colon\mathcal{J}\to\underline{\mathcal{C}at}_1$ be the functor sending an object $j\in\mathcal{J}$ to the slice category $\mathcal{J}_{/j}$. Then the W-weighted limit of a functor $F\colon\mathcal{J}\to\mathcal{C}$ is the *lax limit* of F.

- **Lax limits of morphisms:** Let $n = 2$, $\mathcal{J} = [1]$, then $W = [1]_{/(-)} \colon [1] \to \underline{\mathcal{C}at}_1$ is the diagram in $\mathcal{C}at$ given by
$$[0] = [1]_{/0} \xrightarrow{0} [1] = [1]_{/1}$$
Then the W-weighted limit of a functor $F \colon [1] \to \mathcal{C}$ in $\mathcal{C}at_2$, corresponding to a morphism $f \colon x \to y$ in $\mathcal{C}$, is the *lax limit* of f in $\mathcal{C}$:
$$\begin{tikzcd} \mathrm{laxlim}_{[1]} F \arrow[r] \arrow[dr] & x \arrow[d, "f"] \arrow[dl, phantom, "\Swarrow"] \\ & y \end{tikzcd}$$

Remark 4.3.3. In the case $n = 1$, the weighted limit of a functor $F \colon \mathcal{J} \to \mathcal{C}$ in $\mathcal{C}at_1$ by a functor $W \colon \mathcal{J} \to \mathcal{G}pd$ can be recovered as the limit of the functor $\int_{\mathcal{J}} W \to \mathcal{J} \xrightarrow{F} \mathcal{C}$. For $n > 2$, the object $\int_{\mathcal{J}} W$ is a double $(n-1)$-category (and not an n-category) and so the W-weighted limit of F can not be recovered as a conical n-limit, but only as a "conical double $(n-1)$-limit".

Definition 4.3.4. Let $F \colon \mathcal{J} \to \mathcal{C}$ and $W \colon \mathcal{J}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ be functors in $\mathcal{C}at_n$. A **W-weighted n-colimit of** F is an object $\mathrm{colim}^W_{\mathcal{J}} F$ in $\mathcal{C}$ such that, for all objects $x \in \mathcal{C}$, there is an isomorphism in $\mathcal{C}at_{n-1}$
$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}^W_{\mathcal{J}} F, x) \cong \mathrm{Hom}_{[\mathcal{J}, \underline{\mathcal{C}at}_{n-1}]}(W, \mathrm{Hom}_{\mathcal{C}}(F-, x)),$$
which is n-natural in $x \in \mathcal{C}$. The right-hand side is the functor in $\mathcal{C}at_n$ given by the composite
$$\mathcal{C} \xrightarrow{y_{\mathcal{C}}} [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \xrightarrow{F^{\mathrm{op},*}} [\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \xrightarrow{\mathrm{Hom}_{[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(W,-)} \underline{\mathcal{C}at}_{n-1}.$$

Remark 4.3.5. Let $F \colon \mathcal{J} \to \mathcal{C}$ and $W \colon \mathcal{J}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ be functors in $\mathcal{C}at_n$. We have a canonical isomorphism in $\mathcal{C}$
$$\mathrm{colim}^W_{\mathcal{J}} F \cong \lim^W_{\mathcal{J}^{\mathrm{op}}} F^{\mathrm{op}}.$$

As a consequence of the n-Yoneda lemma, we have that n-(co)limits weighted by representables always exist.

Proposition 4.3.6 (Exercise)**.** *Given a functor $F \colon \mathcal{J} \to \mathcal{C}$ in $\mathcal{C}at_n$ and an object $j \in \mathcal{J}$,*

- *the $\mathrm{Hom}_{\mathcal{J}}(j,-)$-weighted n-limit of F exists, and can be computed as*
$$\lim^{\mathrm{Hom}_{\mathcal{J}}(j,-)}_{\mathcal{J}} F \cong Fj.$$
- *the $\mathrm{Hom}_{\mathcal{J}}(-,j)$-weighted n-colimit of F exists, and can be computed as*
$$\mathrm{colim}^{\mathrm{Hom}_{\mathcal{J}}(-,j)}_{\mathcal{J}} F \cong Fj.$$

Also as a consequence of the n-Yoneda lemma, we have that every n-(co)presheaf can be obtained as a weighted colimit of the Yoneda embedding.

Proposition 4.3.7. *Given a functor $W \colon \mathcal{J}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$, the W-weighted n-colimit of the n-Yoneda embedding $y_{\mathcal{J}} \colon \mathcal{J} \to [\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ exists, and can be computed as*
$$\mathrm{colim}^W_{\mathcal{J}} y_{\mathcal{J}} \cong W.$$

Proof. We prove that the object $W \in [\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ satisfies the universal property of the W-weighted n-colimit of $y_{\mathcal{J}}$. By the n-Yoneda lemma, we have an n-natural isomorphism of functors $\mathcal{J}^{\mathrm{op}} \times [\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$
$$\mathrm{ev} \cong \mathrm{Hom}_{[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(y_{\mathcal{J}}(-), -).$$

By cartesian closedness of $\mathcal{C}at_n$, this corresponds to an n-natural isomorphism of functors $[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \to [\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ in $\mathcal{C}at_n$

$$\mathrm{id}_{[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]} \cong \mathrm{Hom}_{[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(y_{\mathcal{J}}, -).$$

By postcomposing with the functor $\mathrm{Hom}_{[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(W, -) \colon [\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \to \underline{\mathcal{C}at}_{n-1}$, we obtain an n-natural isomorphism of functors $[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$

$$\mathrm{Hom}_{[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(W, -) \cong \mathrm{Hom}_{[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(W, \mathrm{Hom}_{[\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(y_{\mathcal{J}}, -)). \qquad \square$$

By taking $W = \mathrm{cst}[0]$, we get the following corollary.

Corollary 4.3.8. *Given an n-category $\mathcal{J}$, the n-colimit of the n-Yoneda embedding $y_{\mathcal{J}} \colon \mathcal{J} \to [\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ exists, and can be computed as*

$$\mathrm{colim}_{\mathcal{J}} y_{\mathcal{J}} \cong \mathrm{cst}[0].$$

5. Lecture 5: Free cocompletion by all weighted colimits

5.1. Weighted n-limits in $\underline{\mathcal{C}at}_{n-1}$.

We prove that $\underline{\mathcal{C}at}_{n-1}$ has all weighted n-limits, by giving an explicit formula for those.

Definition 5.1.1. Let $\mathcal{C}$ be an n-category and $\mathcal{X} \in \mathcal{C}at_{n-1}$.

- ▷ The **$\mathcal{X}$-cotensor** of an object $c \in \mathcal{C}$ is the n-limit of the functor $c \colon [0] \to \mathcal{C}$ weighted by $\mathcal{X} \colon [0] \to \underline{\mathcal{C}at}_{n-1}$, and is denoted by $c^{\mathcal{X}}$. It satisfies the universal property that, for every object $y \in \mathcal{C}$, there is an isomorphism in $\mathcal{C}at_{n-1}$
$$\mathrm{Hom}_{\mathcal{C}}(y, c^{\mathcal{X}}) \cong \mathrm{Hom}_{\mathcal{C}at_{n-1}}(\mathcal{X}, \mathrm{Hom}_{\mathcal{C}}(y, c))$$
which is n-natural in $y \in \mathcal{C}^{\mathrm{op}}$.
- ▷ The **$\mathcal{X}$-tensor** of an object $c \in \mathcal{C}$ is the n-colimit of the functor $c \colon [0] \to \mathcal{C}$ weighted by $\mathcal{X} \colon [0] \to \underline{\mathcal{C}at}_{n-1}$, and is denoted by $c \otimes \mathcal{X}$. It satisfies the universal property that, for every object $y \in \mathcal{C}$, there is an isomorphism in $\mathcal{C}at_{n-1}$
$$\mathrm{Hom}_{\mathcal{C}}(c \otimes \mathcal{X}, y) \cong \mathrm{Hom}_{\mathcal{C}at_{n-1}}(\mathcal{X}, \mathrm{Hom}_{\mathcal{C}}(c, y))$$
which is n-natural in $y \in \mathcal{C}$.

Example 5.1.2. The following n-categories have all tensors and cotensors by $(n-1)$-categories:

- ▷ the associated n-category $\underline{\mathcal{A}}$ of a tensored, enriched, and cotensored category $\mathcal{A}$;
- ▷ the n-category $\underline{\mathcal{C}at}_{n-1}$;
- ▷ given an object $X \in \mathcal{C}at_{n-1}^{\Delta^{\mathrm{op}}}$, the n-categories $\underline{\mathcal{C}at}_{n-1}^{\Delta^{\mathrm{op}}}{}_{/X}$ and $\underline{\mathcal{D}bl\mathcal{L}\mathcal{F}ib}(X)$;
- ▷ given an n-category $\mathcal{C}$, the functor n-category $[\mathcal{C}, \underline{\mathcal{C}at}_{n-1}] \cong \underline{\mathcal{D}bl\mathcal{L}\mathcal{F}ib}(\mathcal{C})$: given a functor $F \colon \mathcal{C} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$ and $\mathcal{X} \in \mathcal{C}at_{n-1}$, their tensor and cotensor are given by
$$F \otimes \mathcal{X} \colon \mathcal{C} \xrightarrow{F} \underline{\mathcal{C}at}_{n-1} \xrightarrow{- \times \mathcal{X}} \underline{\mathcal{C}at}_{n-1} \quad \text{and} \quad F^{\mathcal{X}} \colon \mathcal{C} \xrightarrow{F} \underline{\mathcal{C}at}_{n-1} \xrightarrow{\mathrm{Hom}_{\mathcal{C}at_{n-1}}(\mathcal{X}, -)} \underline{\mathcal{C}at}_{n-1}.$$

Construction 5.1.3. Let $\mathcal{C}$ be an n-category, $x \in \mathcal{C}$ be an object and $G \colon \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ be a functor in $\mathcal{C}at_n$. Every natural transformation of functors $\mathcal{C}_{\leq 1}^{\mathrm{op}} \to \mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{\mathcal{C}}(-, x) \Rightarrow G$$

determines, by applying the underlying groupoid functor $\mathcal{C}at_{n-1} \to \mathcal{G}pd$, a natural transformation of functors $\mathcal{C}_{\leq 1}^{\mathrm{op}} \to \mathcal{G}pd$

$$\mathrm{Map}_{\mathcal{C}_{\leq 1}}(-, x) \Rightarrow G^{\simeq}.$$

By Yoneda, this natural transformation is determined by an object $\lambda \in Gx$ and, by the n-Yoneda lemma, that object $\lambda \in Gx$ determines an n-natural transformation of functors $\mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$

$$\mathrm{Hom}_{\mathcal{C}}(-,x) \Rightarrow G.$$

Theorem 5.1.4. *Let $F \colon \mathcal{J} \to \underline{\mathcal{C}at}_{n-1}$ and $W \colon \mathcal{J} \to \underline{\mathcal{C}at}_{n-1}$ be functors in $\mathcal{C}at_n$. The W-weighted n-limit of F exists in $\mathcal{C}at_{n-1}$, and can be computed as*

$$\lim\nolimits_{\mathcal{J}}^{W} F \cong \mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(W,F) \quad \left(\cong \mathrm{Hom}_{\underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1}/\mathcal{J}}\left(\textstyle\int_{\mathcal{J}} W, \int_{\mathcal{J}} F\right)\right)$$

Proof. We prove that the $(n-1)$-category $\mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(W,F)$ satisfies the universal property of the W-weighted n-limit of F. There are natural isomorphisms in $\mathcal{C}at_{n-1}$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}at_{n-1}}(\mathcal{X}, \mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(W,F)) &\cong \mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(W,F^{\mathcal{X}}) \\ &\cong \mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(W,\mathrm{Hom}_{\mathcal{C}at_{n-1}}(\mathcal{X},F-))\end{aligned}$$

which can be promoted to an n-natural isomorphism of functors $\underline{\mathcal{C}at}^{\mathrm{op}}_{n-1} \to \underline{\mathcal{C}at}_{n-1}$. $\square$

Taking $W = \mathrm{cst}[0]$, we get:

Corollary 5.1.5. *Let $F \colon \mathcal{J} \to \underline{\mathcal{C}at}_{n-1}$ be a functor in $\mathcal{C}at_n$. The n-limit of F exists in $\mathcal{C}at_{n-1}$, and can be computed as*

$$\lim\nolimits_{\mathcal{J}} F \cong \mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(\mathrm{cst}[0],F) \quad \left(\cong \mathrm{Hom}_{\underline{\mathcal{C}at}^{\Delta^{\mathrm{op}}}_{n-1}/\mathcal{J}}\left(\mathcal{J}, \textstyle\int_{\mathcal{J}} F\right)\right)$$

Proposition 5.1.6. *Given an n-category $\mathcal{C}$ and an object $x \in \mathcal{C}$, the representable functor*

$$\mathrm{Hom}_{\mathcal{C}}(x,-) \colon \mathcal{C} \to \underline{\mathcal{C}at}_{n-1}$$

preserves weighted n-limits.

Proof. Let $F \colon \mathcal{J} \to \mathcal{C}$ and $W \colon \mathcal{J} \to \underline{\mathcal{C}at}_{n-1}$ be functors such that the W-weighted n-limit of F exists in $\mathcal{C}$. Using the Theorem, we have canonical isomorphisms in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{\mathcal{C}}(x, \lim\nolimits_{\mathcal{J}}^{W} F) \cong \mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(W, \mathrm{Hom}_{\mathcal{C}}(x,F-)) \cong \lim\nolimits_{\mathcal{J}}^{W} \mathrm{Hom}_{\mathcal{C}}(x,F-).$$ $\square$

Proposition 5.1.7. *Given an n-category $\mathcal{C}$ and an object $x \in \mathcal{C}$, the representable functor*

$$\mathrm{Hom}_{\mathcal{C}}(-,x) \colon \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$$

preserves weighted n-limits.

Proof. Let $F \colon \mathcal{J} \to \mathcal{C}$ and $W \colon \mathcal{J}^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ be functors such that the W-weighted n-colimit of F exists in $\mathcal{C}$. Using the Theorem, we have canonical isomorphisms in $\mathcal{C}at_{n-1}$

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{J}}^{W} F, x) \cong \mathrm{Hom}_{[\mathcal{J}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]}(W, \mathrm{Hom}_{\mathcal{C}}(F-,x)) \cong \lim\nolimits_{\mathcal{J}^{\mathrm{op}}}^{W} \mathrm{Hom}_{\mathcal{C}}(F-,x)$$ $\square$

5.2. **Weighted n-(co)limits in functors categories.** We now turn to prove that weighted n-(co)limits in functor categories exist whenever the pointwise weighted n-(co)limits do.

Proposition 5.2.1. *Given an n-category $\mathcal{C}$, the n-Yoneda embedding*

$$y_{\mathcal{C}} \colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$$

preserves weighted n-limits.

Proof. Let $F\colon \mathcal{J} \to \mathcal{C}$ and $W\colon \mathcal{J} \to \underline{\mathcal{C}at}_{n-1}$ be functors in $\mathcal{C}at_n$ such that the limit of F exists in $\mathcal{C}$. By definition, we have canonical isomorphisms in $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$

$$\begin{aligned} y_{\mathcal{C}}(\lim^W_{\mathcal{J}} F) &\cong \mathrm{Hom}_{\mathcal{C}}(-, \lim^W_{\mathcal{J}} F) \cong \mathrm{Hom}_{[\mathcal{J}, \underline{\mathcal{C}at}_{n-1}]}(W, \mathrm{Hom}_{\mathcal{C}}(-, F)) \\ &\cong \mathrm{Hom}_{[\mathcal{J}, \underline{\mathcal{C}at}_{n-1}]}(W, (y_{\mathcal{C}})_* F) \cong \lim^W_{\mathcal{J}}((y_{\mathcal{C}})_* F) \end{aligned}$$

where $(y_{\mathcal{C}})_*\colon [\mathcal{J}, \mathcal{C}] \to [\mathcal{J}, [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at_{n-1}]]$ is postcomposition with $y_{\mathcal{C}}$ and the last isomorphism follows from the previous result. ☐

Corollary 5.2.2. *For all n-category $\mathcal{I}$ and $\mathcal{C}$, the postcomposition with the n-Yoneda embedding*

$$(y_{\mathcal{C}})_*\colon [\mathcal{I}, \mathcal{C}] \to [\mathcal{I}, [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at_{n-1}]]$$

preserves pointwise weighted n-limits.

Proposition 5.2.3. *Let $F\colon \mathcal{J} \to [\mathcal{I}, \mathcal{C}]$ and $W\colon \mathcal{J} \to \underline{\mathcal{C}at}_{n-1}$ be functors in $\mathcal{C}at_n$. If for every object $i \in \mathcal{I}$ the W-weighted n-limit of $F(-)(i)\colon \mathcal{J} \to \mathcal{C}$ exists in $\mathcal{C}$, then the W-weighted n-limit of F exists in $[\mathcal{I}, \mathcal{C}]$, and is computed pointwise, i.e., for each object $i \in \mathcal{I}$, we have a canonical isomorphism in $\mathcal{C}$*

$$(\lim^W_{\mathcal{J}} F)(i) \cong \lim^W_{\mathcal{J}}(F(-)(i)).$$

Proof idea. **Case $\mathcal{C} = \underline{\mathcal{C}at}_{n-1}$:** One can prove that the composite

$$\mathcal{I} \xrightarrow{F} [\mathcal{J}, \underline{\mathcal{C}at}_{n-1}] \xrightarrow{\mathrm{Hom}_{[\mathcal{J}, \underline{\mathcal{C}at}_{n-1}]}(W,-)} \underline{\mathcal{C}at}_{n-1}$$

satisfies the universal property of the W-weighted n-limit of F in $[\mathcal{I}, \underline{\mathcal{C}at}_{n-1}]$. Using the formula for weighted n-limits in $\underline{\mathcal{C}at}_{n-1}$, this functor satisfies the desired pointwise condition.

General case: Postcompose F with $(y_{\mathcal{C}})_*\colon [\mathcal{I}, \mathcal{C}] \to [\mathcal{I}, [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]] \cong [\mathcal{I} \times \mathcal{C}^{op}, \underline{\mathcal{C}at}_{n-1}]$. By the first part of the proof and using that all W-weighted n-limits exist in $\underline{\mathcal{C}at}_{n-1}$, the W-weighted n-limit of $(y_{\mathcal{C}})_* F$ exists in $[\mathcal{I}, [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]]$ and so we have a functor in $\mathcal{C}at_n$

$$\lim^W_{\mathcal{J}}((y_{\mathcal{C}})_* F)\colon \mathcal{I} \to [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \tag{1}$$

Since $(y_{\mathcal{C}})_*$ preserves pointwise weighted n-limits, given an object $i \in \mathcal{I}$, we have canonical isomorphisms in $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$

$$(\lim^W_{\mathcal{J}}((y_{\mathcal{C}})_* F))(i) \cong \lim^W_{\mathcal{J}}((y_{\mathcal{C}})_* F(-)(i)) \cong y_{\mathcal{C}}(\lim^W_{\mathcal{J}} F(-)(i))$$

Hence, by corestricting the functor (1) along the n-Yoneda embedding $y_{\mathcal{C}}\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ we obtain a functor in $\mathcal{C}at_n$

$$\lim^{W,\mathrm{pt}}_{\mathcal{J}} F\colon \mathcal{I} \to \mathcal{C}$$

sending an object $i \in \mathcal{I}$ to $\lim^W_{\mathcal{J}}(F(-)(i))$.

We prove that the functor $\lim^{W,\mathrm{pt}}_{\mathcal{J}} F\colon \mathcal{I} \to \mathcal{C}$ satisfies the universal property of the W-weighted n-limit of F in $[\mathcal{I}, \mathcal{C}]$. We have n-natural isomorphisms of functors $[\mathcal{J}, \mathcal{C}]^{\mathrm{op}} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$

$$\begin{aligned} &\mathrm{Hom}_{[\mathcal{I},\mathcal{C}]}(-, \lim^{W,\mathrm{pt}}_{\mathcal{J}} F) \\ &\quad\cong \mathrm{Hom}_{[\mathcal{I},[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]]}((y_{\mathcal{C}})_*(-), (y_{\mathcal{C}})_* \lim^{W,\mathrm{pt}}_{\mathcal{J}} F) && (y_{\mathcal{C}})_* \text{ fully faithful} \\ &\quad\cong \mathrm{Hom}_{[\mathcal{I},[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]]}((y_{\mathcal{C}})_*(-), \lim^{W,\mathrm{pt}}_{\mathcal{J}}((y_{\mathcal{C}})_* F)) && (y_{\mathcal{C}})_* \text{ pres. pt } n\text{-limits} \\ &\quad\cong \mathrm{Hom}_{[\mathcal{I},[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]]}((y_{\mathcal{C}})_*(-), \lim^{W}_{\mathcal{J}}((y_{\mathcal{C}})_* F)) && n\text{-limits in presheaves} \\ &\quad\cong \mathrm{Hom}_{[\mathcal{J}, \underline{\mathcal{C}at}_{n-1}]}(W, \mathrm{Hom}_{[\mathcal{I},[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]]}((y_{\mathcal{C}})_*(-), (y_{\mathcal{C}})_* F)) && \text{definition} \\ &\quad\cong \mathrm{Hom}_{[\mathcal{J}, \underline{\mathcal{C}at}_{n-1}]}(W, \mathrm{Hom}_{[\mathcal{I},\mathcal{C}]}(-, F)) && (y_{\mathcal{C}})_* \text{ fully faithful} \end{aligned}$$

as desired. ☐

Corollary 5.2.4. *Let $\mathcal{C}$ be an n-category with all weighted n-limits. For every n-category $\mathcal{I}$, the n-category $[\mathcal{I},\mathcal{C}]$ also has all weighted n-limits.*

Proposition 5.2.5. *Given an n-category $\mathcal{C}$ with all weighted n-limits, there is a functor in $\mathcal{C}at_n$*

$$\lim_{\mathcal{J}}^{(-)}(-)\colon [\mathcal{J},\underline{\mathcal{C}at}_{n-1}]^{\mathrm{op}}\times[\mathcal{J},\mathcal{C}]\to\mathcal{C}$$

sending a pair $(W\colon\mathcal{J}\to\underline{\mathcal{C}at}_{n-1},F\colon\mathcal{J}\to\mathcal{C})$ to $\lim_{\mathcal{J}}^W F$.

Proof idea. **Case $\mathcal{C}=\underline{\mathcal{C}at}_{n-1}$:** The weighted n-limit functor is given by the hom-functor

$$\mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(-,-)\colon [\mathcal{J},\underline{\mathcal{C}at}_{n-1}]^{\mathrm{op}}\times[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]\to\underline{\mathcal{C}at}_{n-1}.$$

Case $\mathcal{C}=[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]$: The weighted n-limit functor is induced by postcomposing with the functor from the previous case.

General case: The weighted n-limit functor is obtained by coretricting the functor from the previous case along the n-Yoneda embedding. □

Remark 5.2.6. The results can be dualized to colimits by passing to the opposite category $\mathcal{C}^{\mathrm{op}}$.

5.3. **Free cocompletions by all weighted n-colimits.** Finally, we prove that the n-category of n-presheaves over an n-category is its free cocompletion by all weighted n-colimits.

Theorem 5.3.1. *Let $F\colon\mathcal{J}\to\underline{\mathcal{C}at}_{n-1}$ and $W\colon\mathcal{J}^{\mathrm{op}}\to\underline{\mathcal{C}at}_{n-1}$ be functors in $\mathcal{C}at_n$. The W-weighted n-colimit of F exists in $\mathcal{C}at_{n-1}$, and can be computed as*

$$\operatorname{colim}_{\mathcal{J}}^W F\cong\operatorname{colim}_{\Delta^{\mathrm{op}}}\left(\textstyle\int_{\mathcal{J}}W\times_{\mathcal{J}}\int_{\mathcal{J}}F\right).$$

Corollary 5.3.2. *Given an n-category $\mathcal{C}$, the n-category $[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]$ has all weighted n-colimits.*

Using these results, we can prove that weighted n-limits commute with weighted n-limits, and deduce the similar result for colimits.

Proposition 5.3.3. *Let $F\colon\mathcal{J}\to\mathcal{C}$, $V\colon\mathcal{I}^{\mathrm{op}}\to\underline{\mathcal{C}at}_{n-1}$, and $W\colon\mathcal{I}\to[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]$ be functors in $\mathcal{C}at_n$. There is a canonical isomorphism in $\mathcal{C}at_{n-1}$*

$$\lim_{\mathcal{J}}^{\operatorname{colim}_{\mathcal{I}}^V W}F\cong\lim_{\mathcal{I}^{\mathrm{op}}}^V\lim_{\mathcal{J}}^W F.$$

Proof. **Case $\mathcal{C}=\underline{\mathcal{C}at}_{n-1}$:** We have canonical isomorphisms in $\mathcal{C}at_{n-1}$

$$\begin{aligned}\lim_{\mathcal{J}}^{\operatorname{colim}_{\mathcal{I}}^V W}F&\cong\mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(\operatorname{colim}_{\mathcal{I}}^V W,F) && n\text{-limits in }\underline{\mathcal{C}at}_{n-1}\\&\cong\lim_{\mathcal{I}^{\mathrm{op}}}^V\mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(W-,F) && \mathrm{Hom}_{[\mathcal{J},\underline{\mathcal{C}at}_{n-1}]}(-,F)\text{ pres. }n\text{-limits}\\&\cong\lim_{\mathcal{I}^{\mathrm{op}}}^V\lim_{\mathcal{J}}^W F && n\text{-limits in presheaves}\end{aligned}$$

Case $\mathcal{C}=[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]$: Follows from the above using that weighted n-limits in $[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]$ are computed pointwise in $\underline{\mathcal{C}at}_{n-1}$.

General case: We have n-natural isomorphisms of functors $\mathcal{C}^{\mathrm{op}}\to\underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$

$$\begin{aligned}y_{\mathcal{C}}(\lim_{\mathcal{J}}^{\operatorname{colim}_{\mathcal{I}}^V W}F)&\cong\lim_{\mathcal{J}}^{\operatorname{colim}_{\mathcal{I}}^V W}(y_{\mathcal{C}})_*F && y_{\mathcal{C}}\text{ pres. }n\text{-limits}\\&\cong\lim_{\mathcal{I}^{\mathrm{op}}}^V\lim_{\mathcal{J}}^W(y_{\mathcal{C}})_*F && \text{previous case}\\&\cong y_{\mathcal{C}}(\lim_{\mathcal{I}^{\mathrm{op}}}^V\lim_{\mathcal{J}}^W F) && y_{\mathcal{C}}\text{ pres. }n\text{-limits}\end{aligned}$$

So the result follows from the fact that the n-Yoneda embedding $y_{\mathcal{C}}\colon\mathcal{C}\to[\mathcal{C}^{\mathrm{op}},\underline{\mathcal{C}at}_{n-1}]$ is fully faithful. □

We also have the dual version:

Proposition 5.3.4. *Let $F\colon \mathcal{J} \to \mathcal{C}$, $V\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}at_{n-1}$, and $W\colon \mathcal{I}^{\mathrm{op}} \to [\mathcal{J}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ be functors in $\mathcal{C}at_n$. There is a natural isomorphism in $\mathcal{C}at_{n-1}$*

$$\mathrm{colim}_{\mathcal{J}}^{\mathrm{colim}_{\mathcal{I}}^V W} F \cong \mathrm{colim}_{\mathcal{I}}^V \mathrm{colim}_{\mathcal{J}}^W F.$$

Definition 5.3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be n-categories which have all weighted n-colimits. A functor $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$ is **n-cocontinuous** if it preserves all weighted n-colimits.

We denote by $[\mathcal{C}, \mathcal{D}]^{\mathrm{cocont}} \subseteq [\mathcal{C}, \mathcal{D}]$ the full n-subcategory spanned by the n-cocontinuous functors.

Theorem 5.3.6. *Let $\mathcal{C}$ be an n-category. The n-Yoneda embedding*

$$y_{\mathcal{C}}\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$$

exhibits $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ as the free weighted n-cocompletion of $\mathcal{C}$. In other words, the n-category $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ has all weighted n-colimits and, for every n-category $\mathcal{D}$ with all weighted n-colimits, the induced functor

$$y_{\mathcal{C}}^*\colon [[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}], \mathcal{D}]^{\mathrm{cocont}} \to [\mathcal{C}, \mathcal{D}]$$

is an isomorphism in $\mathcal{C}at_n$.

Proof idea. By the above, we know that $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]$ has all weighted n-colimits.

Given an n-category $\mathcal{D}$ with all weighted n-colimits, we construct an inverse of the functor $y_{\mathcal{C}}^*\colon [[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}], \mathcal{D}]^{\mathrm{cocont}} \to [\mathcal{C}, \mathcal{D}]$. The functor in $\mathcal{C}at_n$

$$\mathrm{colim}_{\mathcal{C}}^{(-)}(-)\colon [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \times [\mathcal{C}, \mathcal{D}] \to \mathcal{D}.$$

induces, by cartesian closedness, a functor in $\mathcal{C}at_n$

$$\mathrm{colim}_{\mathcal{C}}^{(-)}(-)\colon [\mathcal{C}, \mathcal{D}] \to [[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}], \mathcal{D}].$$

This functor takes value in the full n-subcategory $[[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}], \mathcal{D}]^{\mathrm{cocont}}$ since, given a functor $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$, its image $\mathrm{colim}_{\mathcal{C}}^{(-)} F\colon [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \to \mathcal{D}$ preserves weighted n-colimits by the previous proposition.

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$, by definition of weighted n-colimits and the n-Yoneda lemma, we have n-natural isomorphisms of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \underline{\mathcal{C}at}_{n-1}$ in $\mathcal{C}at_n$

$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}_{\mathcal{C}}^{y_{\mathcal{C}}(-)} F, -) \cong \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}(y_{\mathcal{C}}(-), \mathrm{Hom}_{\mathcal{D}}(F, -)) \cong \mathrm{Hom}_{\mathcal{D}}(F-, -).$$

By fully faithfulness of $(y_{\mathcal{D}^{\mathrm{op}}})_*\colon [\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}] \to [\mathcal{C}^{\mathrm{op}}, [\mathcal{D}, \underline{\mathcal{C}at}_{n-1}]]$, we have an n-natural isomorphism of functors $\mathcal{C} \to \mathcal{D}$ in $\mathcal{C}at_n$

$$\mathrm{colim}_{\mathcal{C}}^{y_{\mathcal{C}}(-)} F \cong F.$$

Next, given a cocontinuous functor $G\colon [\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \to \mathcal{D}$ in $\mathcal{C}at_n$, since G preserves weighted n-colimits, we have n-natural isomorphisms of functors $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}] \to \mathcal{D}$ in $\mathcal{C}at_n$

$$\mathrm{colim}_{\mathcal{C}}^{(-)} y_{\mathcal{C}}^* G \cong \mathrm{colim}_{\mathcal{C}}^{(-)} G_* y_{\mathcal{C}} \cong G(\mathrm{colim}_{\mathcal{C}}^{(-)} y_{\mathcal{C}}) \cong G(\mathrm{id}_{[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{C}at}_{n-1}]}) \cong G. \qquad \square$$

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