Strongly fusion 2-categories are grouplike

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These slides available at categorified.net/UMass-F2C.pdf

[Etingof – Nikshych – Ostrik]

A multifusion 1-category is

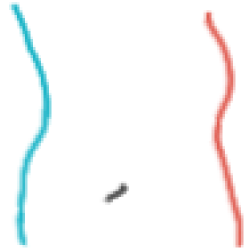
- a finite semisimple 1-category $\mathcal{C}$ over $\mathbb{C}$
  - additive + Karoubi complete
  - $\forall X, Y \in \mathcal{C}$, $\hom(X,Y)$ is a finite dim v-space / $\mathbb{C}$
  - every obj is $\bigoplus$ simples
  - finitely many iso classes of simples

  equivalently, $\exists$ f.d. s.s. $\mathbb{C}$-alg A s.t. $\mathcal{C} \simeq \mathrm{Mod}(A)$.

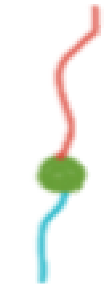
- equipped with a $\otimes$ structure s.t. every object has both left and right duals.

A multifusion category is fusion if $\mathbb{1}$ is simple, i.e. $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \mathbb{C}$.

multifusion categories act on $1+1$ D quantum systems by topological operators. $\longrightarrow$ operators that commute w/ the energy-momentum tensor.

objects X, Y of $\mathcal{C}$ $\longrightarrow$ top. line operators 

tensor product $\otimes$ $\longrightarrow$ fusion of line operators

$f \in \hom(X, Y)$ $\longrightarrow$ junctions 

unit object $\mathbb{1}$ $\longrightarrow$ invisible line operator

$\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ $\longrightarrow$ point operators

Physically important: An action of $\mathcal{C}$ on Q constrains the RG flow / IR limit of Q. It gives Q topological order.

A multifusion 2-category is [Douglas–Reutter]

- a finite semisimple 2-category / $\mathbb{C}$
  - 1-morphisms have both adjoints
  - additive & Karoubi complete
  - $\forall X, Y \in \mathcal{C}$, $\hom(X,Y)$ is a finite s.s. 1-cat / $\mathbb{C}$
  - every object is a $\oplus$ of simples
  - finitely many components

equivalently, $\exists$ multifusion cat $\mathcal{A}$ s.t. $\mathcal{C} \simeq Mod(\mathcal{A})$

"morita equivalence classes of simples"

- equipped with a monoidal str s.t. all objects have duals.

Multifusion 2-categories act on 2+1D quantum systems.

Objects $\rightarrow$ surface operators. 1-morphisms $\rightarrow$ line operators.

$e : X \circlearrowleft$ s.t. $e^2 = e$ $\quad\quad$ $e = fg,\ gf = \mathrm{id}_Y$

A ~~1~~ n-cat is Karoubi complete if every idempotent splits.

A condensation $X \twoheadrightarrow Y$ is
$$\begin{tikzcd} X \arrow[d, bend right, "f"'] \\ Y \arrow[u, bend right, "g"'] \end{tikzcd} \quad \& \quad \underbrace{gf \twoheadrightarrow \mathrm{id}_Y}_{\text{categorifies } gf = \mathrm{id}_Y}$$

A condensation monad is
$$\begin{tikzcd} X \arrow[loop below, "e"] \end{tikzcd} \quad \& \quad \underbrace{e^2 \twoheadrightarrow e}_{\text{categorifies } e^2 = e}$$
& (higher) associativity.

Thm [Douglas–Reutter]: If $\mathcal{C}$ is a ss. n-cat and $X, Y \in \mathcal{C}$ simple and $\hom(X, Y) \neq 0$, then $X \twoheadrightarrow Y$ and $Y \twoheadrightarrow X$.

"Higher-categorical Schur's lemma".

N.B.: Typically X and Y are not isomorphic.

Such X, Y are in the same component.

Recall: A multifusion 1-cat was fusion if $\underbrace{\mathrm{End}_{\mathcal{C}}(\mathbb{1})}_{\substack{\parallel \\ \Omega\mathcal{C}}} = \mathbb{C}$.

What categorifies this notion?

A multifusion 2-cat is:

- fusion if $\mathbb{1}$ is simple, equiv $\underbrace{\mathrm{End}_{\mathrm{End}_{\mathcal{C}}(\mathbb{1})}(\mathrm{id}_{\mathbb{1}})}_{\substack{\parallel \\ \Omega^2\mathcal{C}}} = \mathbb{C}$.
- strongly fusion if $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \mathrm{Vec}$.

In terms of 2+1D quantum systems

fusion $\iff$ no nontrivial top. point operators

strongly fusion $\iff$ no nontriv top. line operators

N.B.: If you have nontrivial point operators, you can build nontrivial line operators via condensation.

Example: Suppose $\mathcal{B}$ is a braided fusion 1-cat.

Then $\mathcal{C} := \mathrm{Mod}(\mathcal{B})$ is a fusion 2-cat.

It is connected: all simples are in the same component.

Every connected fusion 2-cat is of this form, w/ $\mathcal{B} = \mathrm{End}_{\mathcal{C}}(\mathbb{1})$

and the connected component in any fusion 2-cat is of this form.

$\hookrightarrow$ $\oplus$s of simples in the comp. containing $\mathbb{1}$.

Remark: $\mathrm{Mod}(\mathcal{B}) =: \Sigma\mathcal{B}$. $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) =: \Omega\mathcal{B}$.

Notation is justified because there is an adjunction $\Sigma \dashv \Omega$.

The classification of braided fusion 1-categories, and hence of fusion 2-categories, is wild.

It is basically the classification of modular tensor categories and all of their symmetries + anomalies.

For comparison:

Thm [JF-Yu]: Suppose $\mathcal{C}$ is a strongly fusion 2-cat. Then every simple object is invertible.

In other words:

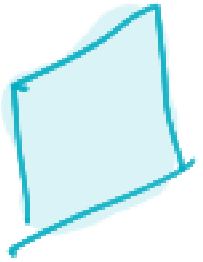
$$\left\{\begin{matrix}\text{strongly fusion}\\ \text{2-cats}\end{matrix}\right\} = \left\{\begin{matrix}\text{finite gps } G\\ \text{plus class in } H^4(BG; \mathbb{C}^\times)\end{matrix}\right\}$$

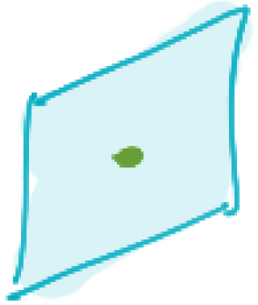
associator / pentagonator data.

We also proved a "super" version: conclusion still holds when $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathrm{SVec}$. (but $H^4 \rightsquigarrow$ more complicated)

Why we care: many classification problems can be reduced to the strongly fusion case.

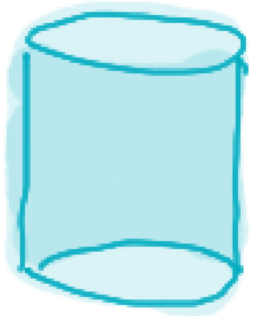
Picture proof (part 1):

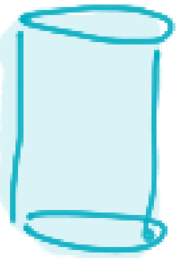
$X \longleftrightarrow$ surface op 

$\mathrm{End}_{\mathrm{End}_e(X)}(\mathrm{id}_X) \longleftrightarrow$ point ops on an X-sheet 

"state-operator correspondence" $\longleftrightarrow$ ways an X cylinder can end

X simple $\longleftrightarrow$ one-dim space of endings

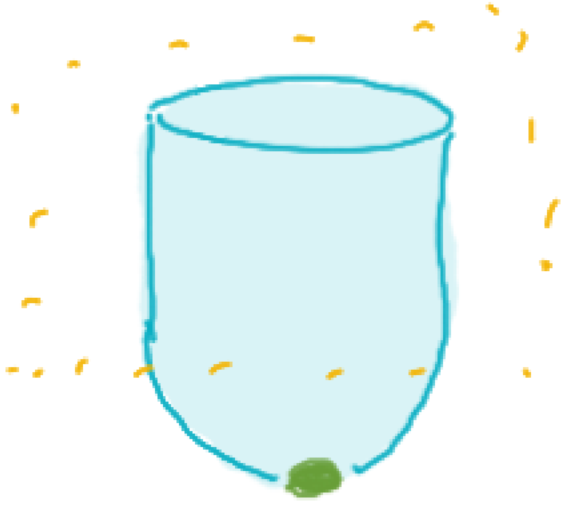
$\mathcal{C}$ strongly fusion $\longleftrightarrow$  $\in \mathrm{Vec}$.

X simple and $\mathcal{C}$ strongly fusion $\longleftrightarrow$  $= \mathbb{C} =$ invisible.

For experts: S^1 has the boundary 2-framing.

$\mathrm{End}_e(\mathbb{1})$

$\cup$

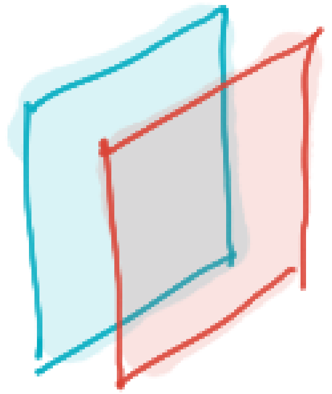
 $=: \int_{S^1} X$

In SVec case:

This cylinder is a s.s. com. alg object in sVec of superdim $=(1,?)$ hence $? = 0$.

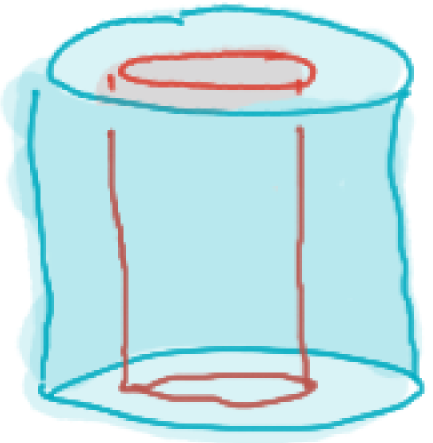
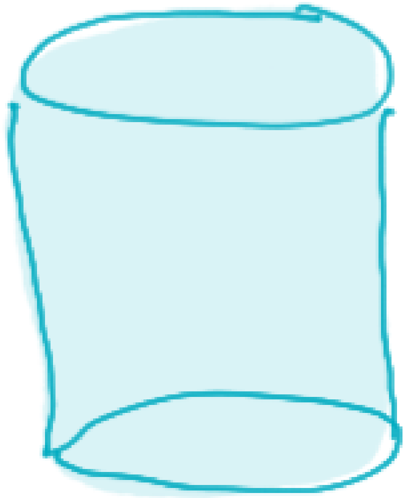
Picture proof (part 2):

Let $X, Y \in \mathcal{C}$ both simple.

$X \otimes Y =$ 

Note: In general, $\int_{S'}$ is not multiplicative.

Point operators on it?

$\int_{S'} X \otimes Y =$  $=$  $= \mathbb{C}$

since Y simple, so  $= \mathbb{C}$

since X simple, so  $= \mathbb{C}$

Thus, if $\mathcal{C}$ is strongly fusion, then simple $\otimes$ simple $=$ simple.

Now take $Y = X^\vee$ (dual object). $\hom(\mathbb{1}, X \otimes X^\vee) \neq 0$,

so $X \otimes X^\vee$ is in the identity component!

But, if $\mathcal{C}$ is strongly fusion, then id comp has only one object. ∎