With six exceptions, nontrivial characters of spheres do not admit gapped boundary conditions

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Based on joint work in progress with David Reutter

these slides:
https://categorified.net/Vanderbilt-2026.pdf







Punchline and Outline

Topological Insulation Theorem [JF–Reutter]: Suppose that X is an n-dimensional invertible insulating quantum material, and assume some reasonable axioms about the words "insulating quantum material." If the invariant $\alpha(X) \in \widehat{\pi}_n^s$ is nontrivial and not one of the six nontrivial Kervaire invariants, then X is a topological insulator: its boundary is necessarily conducting.

Goals for the talk:

- Higher algebra from quantum physics.
- Topological insulation.
- Axioms for the category of insulators.
- Invertible materials and the invariant α.
- Higher algebraic closure.
- Steps in the proof.

Higher algebra for quantum physics

Quantum many-body physics studies functional-analytic systems that extend in spacetime. Examples:

- Quantum field theory (continuous fluctuating fields, local equations of motion)
- Quantum condensed matter (discrete fluctuating fields, quasilocal equations of motion)

Higher algebra studies systems with an infinite hierarchy of algebraic structures. Examples:

- Quasitriangular Hopf algebras (quantum groups, quantum integrable systems)
- L_∞-algebras (deformation quantization, Feynman diagrams)
- Vertex operator algebras (2D conformal field theory, sporadic finite simple groups)
- Cohomology theories (anomaly calculus)

Higher algebra for quantum physics

Fundamental goal of quantum physics: Relate short-distance aka high-energy aka ultraviolet behavior to long-distance aka low-energy aka infrared behavior. Organize possible low-energy systems into "phases". Often: interesting higher algebraic structure emerges in the low energy.

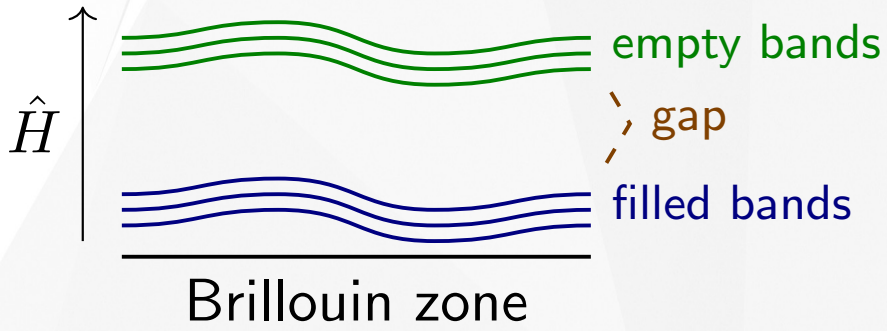
Today I want to tell you about one example: the higher algebra of topological insulators.

Physics[1] **Definition:** A phase of quantum matter is an equivalence class of quantum materials, where two materials are equivalent if you can deform them into each other without triggering a phase transition. Notation: $\{\text{phases of matter}\} = \pi_0\{\text{materials}\}$.

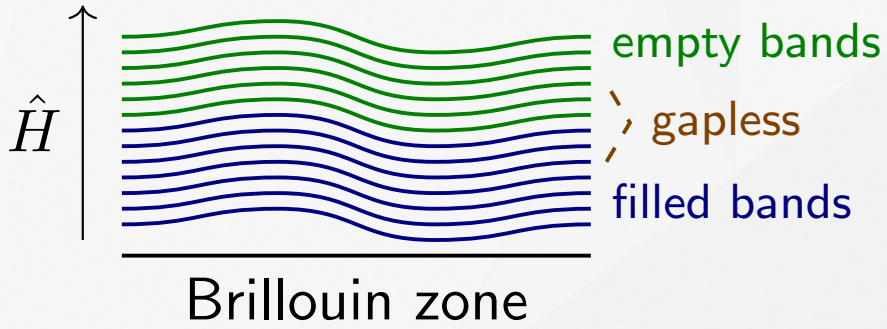
[1] I.e. circular

Topological insulation

A quantum material is insulating when it has a spectral gap: it takes considerable energy to excite it; at low energies it just sits still and does not transmit information/charge/etc.



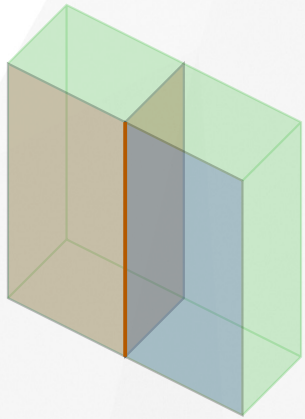
A quantum material is conducting aka metalic when it has continuous spectrum: it is easy to excite it; information/charge/etc readily flows through it.



Topological insulation

A defect in a material can have its own localized excitations. Trivial example: a lower-dimensional material, superimposed on but decoupled from the ambient material. Nontrivial example: ambient material includes a fluctuating field ϕ; defect includes a field whose dynamics depends on the value of ϕ.

An interface is a codimension-one defect separating possibly-different materials. A junction is an interface between defects. A boundary condition is an interface between a material and the vacuum aka trivial material.



Topological insulation

Definition: A topological insulator[2] is an insulator such that all of its boundary conditions are conductors. Some examples[3] in 3+1D:



Bi_2Te_3



Bi_2Se_3



Sb_2Te_3

Typically you expect that you can always destroy conductance by introducing impurities. The term "topological" is because sometimes conductance is unbreakable for topological reasons, for example if the electronic bands have nontrivial Chern numbers.

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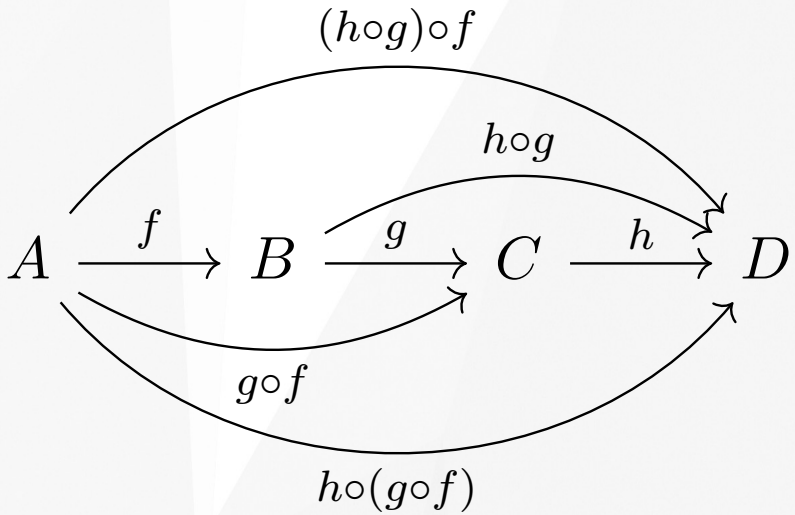
[2] This is a physics term. Mathematicians would never use adjectives this way.

[3] Photos from https://www.hqgraphene.com/.

Axioms for the category of insulators

A category has some set or topological space of "objects" aka "states" aka "systems" and some sets or spaces of "morphisms" aka "transitions" between them, and an associative composition law: $(h \circ g) \circ f = h \circ (g \circ f)$.



Modern categories allow associativity up to (coherent) higher homotopy. Needed for physics: reassociating typically preserves phase equivalence classes but changes the microscopic details.

Axioms for the category of insulators

Axiom -1: Words like “insulator” and “conductor” are analytically well-defined. Justification: Human experience should match mathematical reality.

Axiom 0: Interfaces between insulators can be placed side-by-side. The precise distance does not affect the low-energy behavior. Justification: You cannot measure the distance if you cannot excite the system and measure its resonance.



Axioms for the category of insulators

Since interfaces can be moved without changing the low-energy behavior, they can be fused: move them infinitely close together. Write $\mathcal{T}^n$ for the collection of all nD insulating materials, and for $X, Y \in \mathcal{T}^n$, write $\hom_{\mathcal{T}^n}(X, Y)$ for the collection of insulating interfaces between them. Axiom 0 says: $\mathcal{T}^n$ is a category.

Write $1 \in \mathcal{T}^n$ for the trivial "vacuum" material. Definition: $X \in \mathcal{T}^n$ is a topological insulator when $\mathcal{T}^n(1, X) = \emptyset$.

Axiom 1: $\hom_{\mathcal{T}^n}(1, 1) \cong \mathcal{T}^{n-1}$. In other words, $\mathcal{T}^\bullet$ is a categorical spectrum. Justification: A vacuum-to-vacuum defect should be the same thing as a material one dimension lower.

Axiom 2: $\mathcal{T}^0 = \mathbb{C}$. Justification: Expect $\mathcal{T}^1 \approx \mathbf{Hilb}$, in which case $\mathcal{T}^0 = \hom_{\mathcal{T}^1}(1, 1) = \hom_{\mathbf{Hilb}}(\mathbb{C}^1, \mathbb{C}^1) = \mathbb{C}$.

Axioms for the category of insulators

Axiom 3: Defects and interfaces can be bent around:



In other words, $\mathcal{T}^n$ is rigid[4].

Axiom 4: For any 1D junction between defects (between . . .), the associative algebra of local operators along it is finite-dimensional semisimple. Justification: Finite-dimensionality is reasonably easy to argue. And surely all algebras in physics are C*-algebras.

[4] Yes, "rigid" means things can be bent flexibly. I didn't pick the name.

Invertible materials and the invariant α

Definition: An insulator X is invertible if there is another insulator $\overline{X}$ such that the fusion $X \cdot \overline{X}$ is in the same phase as 1. Write $\mathrm{Inv}(\mathcal{T}^n) \subset \mathcal{T}^n$ for the space of invertible insulators, and $\pi_0 \mathrm{Inv}(\mathcal{T}^n)$ for the set of invertible insulators up to phase.

It is a formal consequence of Axiom 1 that $\mathrm{Inv}(\mathcal{T}^n)$ is a homotopical abelian group[5] and that $\mathrm{Inv}(\mathcal{T}^{n-1}) \simeq \{\text{paths } 1 \rightsquigarrow 1 \text{ in the space } \mathrm{Inv}(\mathcal{T}^n)\}$. Axiom 2 implies that $\mathrm{Inv}(\mathcal{T}^0) = \mathbb{C}^\times$.

[5] The fancy way to say this: $\mathrm{Inv}(\mathcal{T}^\bullet)$ is a spectrum.

Invertible materials and the invariant α

Deep facts:

- There is a universal sequence of abelian groups π_n^s, finite for $n \geq 1$, which are well-studied but not fully understood. They go by many names, the most common being the (stable) homotopy groups of spheres. They have complicated number-theoretic properties.
- Suppose $\mathcal{I}^\bullet$ is any homotopical abelian group with $\mathcal{I}^0 = \mathbb{C}^\times$, e.g. $\mathcal{I}^\bullet = \mathrm{Inv}(\mathcal{T}^\bullet)$. Set $\widehat{\pi_n^s} = \hom(\pi_n^s, \mathbb{C}^\times)$ to be the groups of characters of spheres. For each n there is a canonical homomorphism[6]

$$\alpha : \pi_0(\mathcal{I}^n) \to \widehat{\pi_n^s}.$$

In the case of $\mathcal{I}^\bullet = \mathrm{Inv}(\mathcal{T}^\bullet)$, $\alpha(X)$ measures the exchange statistics of X: the factors you pick up when you permute multiple copies of X.

[6] Indeed, $\bigoplus \pi_\bullet^s$ is a graded ring, and for any homotopical abelian group $\mathcal{I}^\bullet$, $\bigoplus \pi_\bullet^s$ acts canonically on the graded abelian group $\bigoplus \pi_0(\mathcal{I}^{-\bullet})$.

Higher algebraic closure

A digression. The complex numbers are fake. We put it in the name: imaginary numbers. Actual measurements take positive real values. Imaginary numbers are just pairs of real numbers that with funny sign rules. Why do we use them? Familiarity, and:

Hilbert Nullstellensatz, Gelfand duality: Every not-too-large commutative algebra A has a commutative algebra map $A \to \mathbb{C}$.

Koszul's funny sign rules in homological algebra are similarly fake. We put it in the name: super algebra. And yet they are the basis of Fermi-Dirac statistics and Grassmann variables. Why do we use them? Familiarity, and:

Deligne existence of super fibre functors, Doplicher–Roberts reconstruction: Every not-too-large commutative tensor category A has a commutative tensor category map $A \to \{\mathbb{Z}/2\mathbb{Z}\text{-graded vector spaces with the Koszul sign rule}\}$.

Higher algebraic closure

In other words, super vector spaces are algebraically closed in the same way that $\mathbb{C}$ is. Together with David Reutter, I have spent many years understanding the subtleties of algebraic closure in higher-algebraic, and in particular higher-categorical, settings.

Systems satisfying Axioms 0–4 are such a setting: we think of $\mathcal{T}^\bullet$ as a higher commutative ring. Using ideas from Galois theory and in particular Kummer theory, we can show:

Theorem [JF–Reutter]:

- There is a universal, in the sense of algebraic closure, categorical spectrum $\overline{\mathcal{T}^\bullet}$ satisfying Axioms 0–4.
- For $\overline{\mathcal{T}^\bullet}$, the map $\alpha : \pi_0 \operatorname{Inv}(\overline{\mathcal{T}^n}) \to \widehat{\pi_n^s}$ is an isomorphism.

Steps in the proof

Topological Insulation Theorem [JF–Reutter]: Let $\mathcal{T}^\bullet$ satisfy Axioms 0–4, and $X \in \mathrm{Inv}(\mathcal{T}^n)$. If $\alpha(X) \in \widehat{\pi_n^s}$ is nontrivial and not Kervaire, then $\hom_{\mathcal{T}^n}(1, X) = 0$.

Proof steps:

- It suffices to work with the algebraically closed $\overline{\mathcal{T}^\bullet}$.
- Use ideas from quantum (aka "generalized" aka "noninvertible") symmetry to give a simplification algorithm: any element of $\hom_{\mathcal{T}^n}(1, X)$ can be simplified to one describable by abelian groups and quadratic forms.
- If $\hom_{\overline{\mathcal{T}^n}}(1, X) \neq 0$, then any choice of abelian element gives a formula for $\alpha(X)$ in terms of Gauss sums.
- Use results from lattice theory to show that most formulas in terms of Gauss sums evaluate to the trivial element of $\widehat{\pi_n^s}$.

Steps in the proof

Indeed, every finite abelian group factors as a product of groups of prime-power order, and a further simplification reduces abelian p-groups to either trivial or $\mathbb{Z}/p\mathbb{Z}$. When p is odd, the invariants from $\mathbb{Z}/p\mathbb{Z}$ are trivial. The Kervaire invariants are the ones with Gauss-sum formulas using $\mathbb{Z}/2\mathbb{Z}$.

Theorem [Browder, Mahowald, Tangora, Barratt, Jones, Hill, Hopkins, Ravenel, Lin, Wang, Xu]: The Kervaire invariant is trivial except when $n = 2$, 6, 14, 30, 62, or 126.

Example: Massive free fermions are always insulating and invertible. The 2D massive free fermion realizes the 2D Kervaire invariant. It does have an insulating boundary condition. The 3D massive free fermion has a nontrivial quantum Hall conductance. This forces all boundary conditions to be conducting: it is a topological insulator.


THANK YOU!

The universal groups π_n^s

Theorem [Barratt–Priddy–Quillen, Pontryagin–Thom]: For each n, the following abelian groups are equal and called "π_n^s":

- Homotopy groups of the free homotopical abelian group on one generator.
- Stable homotopy groups of spheres: $\pi_n^s = \pi_{n+k}S^k$, $k \gg n$.
- Homology groups of symmetric groups ($k \gg 0$):
  - $\pi_1^s = \mathrm{H}_1(\mathrm{Sym}_k) = \mathrm{Sym}_k/[\mathrm{Sym}_k, \mathrm{Sym}_k]$. Alternating group $\mathrm{Alt}_k := \ker(\mathrm{Sym}_k \to \mathrm{H}_1\mathrm{Sym}_k)$ trivializes H_1.
  - $\pi_2^s = \mathrm{H}_2(\mathrm{Alt}_k)$ is the Schur multiplier of Alt_k. The universal cover $\widetilde{\mathrm{Alt}}_k \to \mathrm{Alt}_k$ trivializes H_2.
  - $\pi_3^s = \mathrm{H}_3(\widetilde{\mathrm{Alt}}_k)$.
  - ...
- Bordism groups of manifolds: $\pi_n^s = \{$stably-framed[7] n-manifolds$\}$/stably-framed cobordism.

[7] A manifold is stably-framed when all of its characteristic classes vanish.