Semisimple higher categories

WHCGP, 26 July 2021

Theo Johnson-Freyd (Dalhousie & Perimeter)

Based on jt work in progress w David Reutter

These slides available at categorified.net/WHCGP.pdf

The Definition

"weak n-category" aka "(n,n)-category"

Douglas–Reutter 2018

A linear n-category $\mathcal{C}$ is semisimple when it is:

- Karoubian: $\mathcal{C}$ has $\oplus$s and every n-idempotent has an image.
- locally semisimple: $\forall X \in \mathcal{C}$, $\Omega_X \mathcal{C} := \mathrm{End}(X)$ is a semisimple ~~algebra~~ monoidal $(n-1)$-category.
- rigid: every k-morphism, $0<k<n$, has both adjoints.

An n-idempotent is like an idempotent except instead of asking for a coherent isomorphism $p^2 \simeq p$, ask for coherent maps $p^2 \rightleftarrows p$ realizing p as the image of an $(n-1)$-idempotent supported by p^2.

higher morphisms, e.g. associativity

$$\begin{tikzcd} & p^2 \arrow[dr] & \\ p^3 \arrow[ur] \arrow[dr] & & p \\ & p^2 \arrow[ur] & \end{tikzcd}$$

JF–Gaiotto 2019

Examples

Over a field $\mathbb{K}$ of char $\neq 2$,

$$\mathrm{Rep}(\mathbb{Z}/2) = \left\{ \begin{matrix} \circlearrowleft^{\mathbb{K}} & & \circlearrowleft^{\mathbb{K}} \\ \bullet & & \bullet \\ \text{trivial rep} & & \text{sign rep} \end{matrix} \right\}$$ and $\oplus$s.

$$\mathrm{Rep}(\mathbb{Z}/2) \overset{\shortparallel}{} \mathrm{mod}\left(\mathbb{C}[x]/x^2=1\right)$$

If $\mathbb{K} = \overline{\mathbb{K}} \overset{=}{} \mathbb{C}$, every s.s. 1-cat feels like just a set.

More precisely (accounting for autos): a set with a map to $K(\mathbb{K}^\times, 2)$.

$$2\mathrm{Rep}(\mathbb{Z}/2) = \mathrm{Rep}(\mathbb{Z}/2) \circlearrowright \underset{\text{triv rep}}{\bullet} \xleftrightarrow{\mathrm{Vec}} \underset{\text{rank-1 free module}}{\bullet} \circlearrowleft \mathrm{Vec}[\mathbb{Z}/2] \cong \mathrm{Rep}(\mathbb{Z}/2)$$

$2\mathrm{Mod}(\mathrm{Vec}[\mathbb{Z}/2])$

v is a noninvertible 1-morphism between simple objects.

$=$ 

and $\oplus$s.

$$x^2 \cong 1$$
$$y^2 \cong 1$$
$$v^* v \cong 1 \oplus x$$
$$v v^* \cong 1 \oplus y$$

e.g. $p = 1 \oplus x$ $\quad p^2 = 1 \oplus 1 \oplus x \oplus x \supset 1 \oplus x = p$

Higher Schur's lemma

[Douglas – Reutter 2018]

$X \in \mathcal{C}$ is simple iff all nonzero $X \to Y$ are faithful
iff X is indecomposable
iff $\Omega^n_X \mathcal{C} = \mathbb{K}$.

↑ some s.s. n-cat over $\mathbb{K} = \overline{\mathbb{K}}$.

1-cat Schur:
any nonzero map between simples is an iso.

Proposition:
- If X is simple and $Y \to X$ is not zero, then X is the image of an n-idempotent at Y.
- If X, Y, Z all simple and $X \xrightarrow{\neq 0} Y \xrightarrow{\neq 0} Z$, then $X \xrightarrow{\neq 0} Z$.

Simple X, Y are Schur connected if $\hom(X, Y) \neq 0$.

The Schur components of $\mathcal{C}$ are

$$\pi_0 \mathcal{C} = \text{simples in } \mathcal{C} \,/\, \text{Schur connectivity}.$$

E.g.
$\pi_0\, 2\mathrm{Rep}(\mathbb{Z}/2) = *$.

Higher homotopy sets

A s.s. n-cat $\mathcal{C}$ is pointed connected when $\pi_0 \mathcal{C} = \{*\}$ and you choose a simple object $X \in \mathcal{C}$. Then the $\otimes$ s.s. $(n-1)$-cat $\Omega\mathcal{C} = \mathrm{End}_{\mathcal{C}}(X)$ determines $\mathcal{C}$.

Define $\pi_K \mathcal{C} := \pi_0 \overbrace{\Omega_X^K \mathcal{C}}^{\text{some } \otimes \text{ s.s. } (n-K)\text{-cat}}$, $K \leq n-1$. $\pi_n \mathcal{C} \sim \mathbb{K}$.

Warning: Different choices of X can give non-iso sets π_K. If $X \neq Y$, then $\Omega_X \mathcal{C}$ and $\Omega_Y \mathcal{C}$ will be Morita equiv, but typically not isomorphic.

Warning / feature: π_K is typically <u>not</u> a gp. It is the basis for a ring.

$\mathcal{C}$ is finite when $|\pi_K \mathcal{C}| < \infty \ \forall K$.

Quantum homotopy types

Suppose $\mathcal{X}$ is a finite homotopy type with a map $\alpha : \mathcal{X} \to K(\mathbb{K}^\times, n+1)$.

i.e. $|\pi_k \mathcal{X}| < \infty$ and $\pi_{\gg 0} \mathcal{X} = \{*\}$.

i.e. $[\alpha] \in H^{n+1}(\mathcal{X}; \mathbb{K}^\times)$.

If $\operatorname{char} \mathbb{K} = 0$, can "linearize" $(\mathcal{X}, \alpha)$ to a finite s.s. n-category $\mathcal{C}$.

If $\pi_{\geq n} \mathcal{X} = \{*\}$, then $\pi_k \mathcal{C} \cong \pi_k \mathcal{X}$.

i.e. if $\mathcal{X}$ is an $(n-1)$-type

if we pick a basepoint in $\mathcal{X}$ and use it as the basepoint in $\mathcal{C}$.

Every (finite) s.s. n-category feels like this but for a "quantum" $(n-1)$-type $\mathcal{X}$, where cells attach to "superpositions" of compositions of lower-dim cells.

$\oplus$s and images of higher idempotents.

Skein theory aka embedded cobordism hypothesis

Any invertible k-morphism X can be inserted along $\Sigma^{n-k} \hookrightarrow \mathbb{R}^n$ provided Σ is normally framed: $N_\Sigma \overset{\sim}{\cong} \underline{\mathbb{R}^k}$

triv. bundle

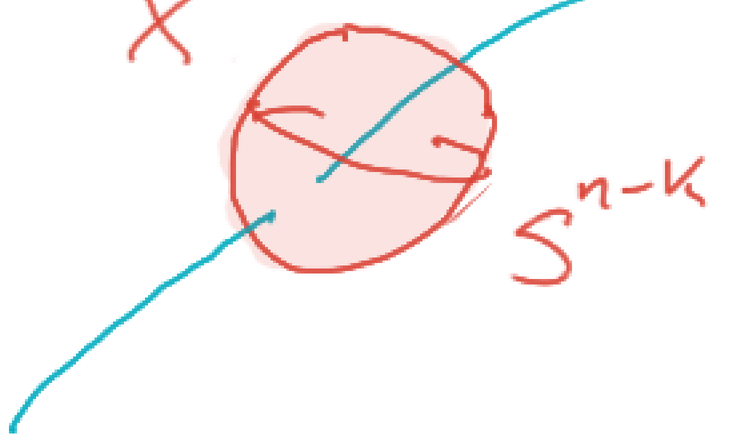
normal bundle

E.g.:

X

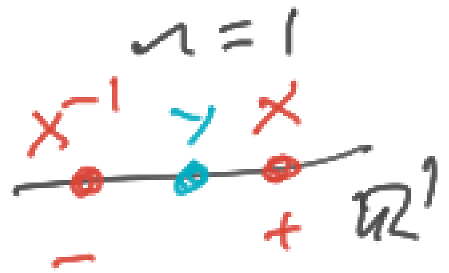
$\subset \mathbb{R}^n$

Y any morphism



S^{n-k} with its standard normal framing as ∂D^{n-k+1}

Is the conjugation action of X on Y.



This is why/how invertible extended top'l operators act by higher-form symmetries.

Extra dualizability

c.f. §4 of Curie's TFT

If our k-morphism X is not invertible, cobordism hyp. says Σ^{n-k} also needs a compatible tangential framing $\left(T_\Sigma \overset{\tau}{\cong} \underline{\mathbb{R}}^{n-k},\ \tau\oplus\nu =: id\right)$

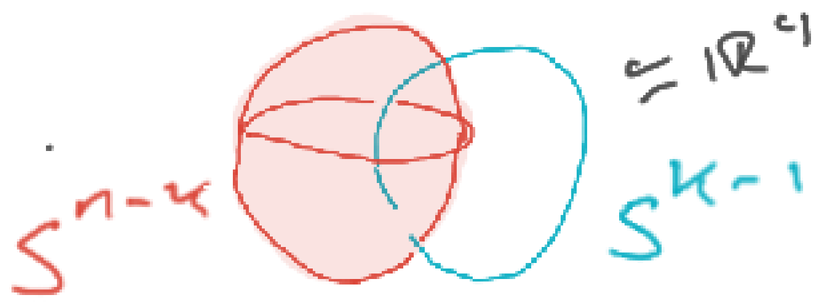
The standard normal framing on S^{n-k} usually does not have a compatible tangential framing.

Theorem: If X is simple and $\mathcal{C}$ is s.s., then there is nevertheless a canonical way to insert it along S^{n-k} as if it were invertible.

This defines a conjugation action by generalized aka categorical aka noninvertible symmetries.

S-matrix aka Whitehead bracket

Suppose $X \in \Omega^k \mathcal{C}$, $Y \in \Omega^{n-k+1} \mathcal{C}$ both simple.

Insert along linked spheres. 

Get a number $\in \mathbb{K}$.

Lemma: This number depends on only the Schur components.

Defines the S-matrix $\tilde{S}_{k,n-k+1} : \pi_k \mathcal{C} \times \pi_{n-k+1} \mathcal{C} \to \mathbb{K} = \pi_n \mathcal{C}$.

In topology, this is Whitehead's lie bracket on $\pi_\bullet \mathcal{X}$.

Example: Let $\mathcal{C}$ = operators in nD Dijkgraaf-Witten theory for a finite gp G. Then $\pi_k \mathcal{C} = \{*\}$ except
$\pi_2 \mathcal{C} = \{\text{conj. classes in } G\}$, $\pi_{n-1} \mathcal{C} = \{\text{irreps of } G\}$
and $\tilde{S}_{2,n-1}$ = character table of G.

Nondegeneracy of the S-matrix

known but not easy when $n=2$. uses a certain $\mathbb{N}$-valued matrix has pos eigenvalue

Conjecture: Over a field of characteristic zero, the sym $\otimes$ $(n+1)$-cat {finite s.s. n-cats} is s.s. and all objects are fully dualizable.

Theorem: Let $\mathcal{C}$ be a f.d. finite s.s. n-category. Then $\mathcal{C}$ is invertible iff $\pi_0 \mathcal{C} = \{*\}$ and the matrix $\tilde{S}_{k,n-k+1}$ is nondegen $\forall k$.

Can test at any simple $x \in \mathcal{C}$.

Application: ($\mathcal{C}$ invertible n-cat, $x \in \mathcal{C}$) defines an anomalous nD TQFT.

This nondegeneracy is a central tool in the classification.