Enriching Yoneda

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Student QFT Conference
11 December 2007

I normally like to avoid all foundational nonsense when I think about abstract nonsense. Everyone knows that the set theorists say that CAT is not an object in CAT, and we almost never care. But stating Yoneda's lemma requires carefully constructing the functor category $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$, which even for normal categories (enriched over SET) is not itself (usually) a category enriched over SET. For example, let $\mathcal{C}$ be the trivial category of cardinalities and only identity morphisms, and let $\mathcal{D}$ be the category of cardinalities and injections. Then we have two immediate functors $\mathcal{C} \to \mathcal{C} \times \mathcal{D}$: we can send $F : X \mapsto (X, 1)$ and $G : X \mapsto (X, X)$. But then the only data in a natural transformation from $F \Rightarrow G$ is a choice of function $1 \to X$ for each X; the collection of all such choice functions is clearly too large to be a set. So when we try to define Yoneda's lemma for enriched categories, we should expect to have to enlarge our categories, just as we have to move to a larger universe when trying to make functor categories over SET.

But before I start going into these details, let me remind you of the setup. We have a particular favorite closed monoidal[1] category $\mathcal{A}$, for example ABGP or SVECT, with monoidal product $\otimes$, functorial in each variable. Being closed means that $\mathrm{Hom}(X, Y) \in \mathcal{A}$ for $X, Y \in \mathcal{A}$. Of course, it's also a category; I will use $\mathrm{Hom}_0(X, Y) = \mathrm{Hom}_{\mathcal{A}_0}(X, Y)$ when I mean the *set* of morphisms, whereas the unmarked $\mathrm{Hom}(-, -) = \mathrm{Hom}_{\mathcal{A}}(-, -)$ takes values in $\mathcal{A}$. Occasionally, I will call $\mathrm{Hom}_0(-, -)$ the "reduced Hom".

Moreover, since $\mathcal{A}$ is monoidal, it has a unit object 1. Then we have a functor $\mathrm{Hom}_0(1, -) : \mathcal{A} \to \mathrm{SET}$, which deserves to be called "forget", because it exactly picks out just the "reduced points". I will occasionally use X_0 for $\mathrm{Hom}_0(1, X)$, and $\mathcal{A}_0$ for the image $\mathrm{Hom}_0(1, \mathcal{A}) \subseteq \mathrm{SET}$. We also have an adjunction[2]

$$\mathrm{adj} : \mathrm{Hom}_0(X, \mathrm{Hom}(Y, Z)) \cong \mathrm{Hom}_0(Y \otimes X, Z)$$

*The material for this talk is lifted almost entirely from G.M. Kelly, *Basic Concepts of Enriched Category Theory*, Cambridge University Press, 2005.

[1] I will assume all my monoidal structures are strong, so that I will not write the associators. For example, when I write SET, I really mean the skeleton: use the Axiom of Choice to pick one set of each cardinality.

[2] We need the adjunction to "switch the order" like this in order to get composition without a braiding.

which justifies the notation:

$$\mathrm{Hom}_0(1, \mathrm{Hom}(X, Y)) \cong \mathrm{Hom}_0(X \otimes 1, Y) = \mathrm{Hom}_0(X, Y).$$

Of course, Yoneda and the adjunction give

$$\mathrm{Hom}(1, Y) \cong Y$$

Moreover, the standard Yoneda's lemma for $\mathcal{A}$ also gives us

$$\mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \cong \mathrm{Hom}(Y \otimes X, Z).$$

Because of our forgetful functor, we can talk about (reduced) points in $\mathcal{A}$-objects, and (reduced) morphisms. But we probably shouldn't. $\mathrm{Hom}_0(1, -)$ is regularly not monoidal — e.g. it does not take $\otimes$ in AbGp to $\times$ in Set — and almost never full, since Set has less structure and hence more morphisms, and sometimes not even faithful: for example, $\mathrm{Hom}_0(1, -) : \textsc{SVect} \to \textsc{Set}$ preserves only the even vectors in a super vector space, and hence only the even morphisms.

In any case, let's say we have an $\mathcal{A}$-enriched category ("A-category") $\mathcal{C}$. I.e. we have a collection of objects, and for any pair of objects X, Y we have an $\mathcal{A}$-object $\mathrm{Hom}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathcal{A}$. Of course, $\mathcal{C}$ is also a Set-enriched category: by hitting $\mathrm{Hom}(X, Y)$ with the forgetful functor we can define a category $\mathcal{C}_0$ with the same objects as $\mathcal{C}$ and Hom-sets given by

$$\mathrm{Hom}_0(X, Y) = \mathrm{Hom}_{\mathcal{C}_0}(X, Y) = \mathrm{Hom}_0(1, \mathrm{Hom}(X, Y)) \in \textsc{Set}.$$

To be an $\mathcal{A}$-category also requires that we have a composition law

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\mu} \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

satisfying associativity:

$$\begin{array}{ccc} \mathrm{Hom}(W, X) \otimes \mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(Y, Z) & \xrightarrow{\mu \otimes \mathrm{id}} & \mathrm{Hom}(W, Y) \otimes \mathrm{Hom}(Y, Z) \\ \downarrow \mathrm{id} \otimes \mu & \# & \downarrow \mu \\ \mathrm{Hom}(W, X) \otimes \mathrm{Hom}(X, Z) & \xrightarrow{\mu} & \mathrm{Hom}(W, Z) \end{array}$$

For example, $\mathcal{A}$ is an $\mathcal{A}$-category: we already discussed the (inner) Hom; the composition $\mathrm{Hom}_{\mathcal{A}}(X, Y) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ comes from the composition in $\mathrm{Hom}_0(-, -)$ and the adjunction: we have an evaluation map

$$\mathrm{id} \in \mathrm{Hom}_0(\mathrm{Hom}_{\mathcal{A}}(X, Y), \mathrm{Hom}_{\mathcal{A}}(X, Y)) \xmapsto{\mathrm{adj}} \mathrm{ev} \in \mathrm{Hom}_0(X \otimes \mathrm{Hom}_{\mathcal{A}}(X, Y), Y)$$

and thus composition:

$$X \otimes \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) \overset{\mathrm{ev}\otimes\mathrm{id}}{\longrightarrow} Y \otimes \mathrm{Hom}(Y,Z) \overset{\mathrm{ev}}{\longrightarrow} Z$$

$$\Downarrow \mathrm{adj}$$

$$\mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) \overset{\circ}{\longrightarrow} \mathrm{Hom}(X,Z)$$

I should also, of course, mention identities, which should be in the reduced part, i.e. an identity is an arrow $1 \to \mathrm{Hom}(X,X)$ satisfying certain diagrams. But for $\mathcal{A}$ as an $\mathcal{A}$-category, $\mathrm{id} \in \mathrm{Hom}_0(X,X)$ gives us exactly such an arrow under the adjunction.

As a second example of an $\mathcal{A}$-enriched category, if $\mathcal{C}$ is an $\mathcal{A}$-category, we might hope to make a category $\mathcal{C}^{\mathrm{op}}$ with the same objects, and with $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$. Composition, however, fails unless $\mathcal{A}$ is braided:

$$\mathrm{Hom}_{\mathrm{op}}(X,Y) \otimes \mathrm{Hom}_{\mathrm{op}}(Y,Z) = \mathrm{Hom}(Y,X) \otimes \mathrm{Hom}(Z,Y) \overset{\mathrm{braid}}{\longrightarrow}$$

$$\mathrm{Hom}(Z,Y) \otimes \mathrm{Hom}(Y,X) \overset{\mu}{\longrightarrow} \mathrm{Hom}(Z,X) = \mathrm{Hom}_{\mathrm{op}}(X,Z)$$

A third example: if $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{A}$-categories, then so is $\mathcal{C}\otimes\mathcal{D}$ with objects being ordered pairs of objects, and $\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((X_{\mathcal{C}}, X_{\mathcal{D}}), (Y_{\mathcal{C}}, Y_{\mathcal{D}})) = \mathrm{Hom}_{\mathcal{C}}(X_{\mathcal{C}}, Y_{\mathcal{C}}) \otimes \mathrm{Hom}_{\mathcal{D}}(Y_{\mathcal{C}}, Y_{\mathcal{D}})$.

Another comment is due about the "forgetful functor" $\mathrm{Hom}_0(1,-)$. The composition law in an $\mathcal{A}$-category $\mathcal{C}$ can be written, via the adjunction, as the morphism "post-compose" in

$$\mathrm{Hom}_0(\mathrm{Hom}_{\mathcal{C}}(Y,Z), \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(X,Y), \mathrm{Hom}_{\mathcal{C}}(X,Z))).$$

But applying $\mathrm{Hom}_0(1,-)$ to this equation gives a function

$$\mathrm{Hom}_0(1, \mathrm{Hom}_{\mathcal{C}}(Y,Z)) = \mathrm{Hom}_{\mathcal{C}_0}(Y,Z) \to \mathrm{Hom}_0(\mathrm{Hom}_{\mathcal{C}_0}(X,Y), \mathrm{Hom}_{\mathcal{C}_0}(X,Z))$$

This is to say that any $\mathcal{A}$-category is also a SET-category.

In any case, what is an $\mathcal{A}$-functor of $\mathcal{A}$-categories $F : \mathcal{C} \to \mathcal{D}$? It should be

- An *assignment* from objects to objects: $X \overset{F}{\mapsto} F(X)$.
- An *arrow* (in $\mathcal{A}$) from Hom-objects to Hom-objects: $\mathrm{Hom}_{\mathcal{C}}(X,Y) \overset{F}{\to} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, i.e. an element of the set $\mathrm{Hom}_0(\mathrm{Hom}_{\mathcal{C}}(X,Y), \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)))$. This should preserve identities and compositions, i.e. certain natural squares must commute.

For example, $\mathrm{Hom}_{\mathcal{C}}(W,-)$ is a functor from $\mathcal{C} \to \mathcal{A}$ for any $W \in \mathcal{C}$, where it acts on morphisms by taking the adjunction of the composition law in $\mathcal{C}$. Of course, by the adjunction, $\mathrm{Hom}_{\mathcal{A}}(1,-)$ is the identity functor on $\mathcal{A}$. Functor composition is obviously well-defined (since we can glue commutative squares) and associative, so the $\mathcal{A}$-categories form a category $\mathcal{A}$CAT with $\mathrm{Hom}_{\mathcal{A}\mathrm{CAT}}(\mathcal{C},\mathcal{D}) = \mathrm{Fun}_{\mathcal{A}}(\mathcal{C},\mathcal{D})$. Our first goal, then, is to enrich this into an $\mathcal{A}$-2-category (i.e. make the Homs into $\mathcal{A}$-categories), and make sense of "fully faithful". Once we have done so, we will be able to state (and prove) Yoneda's lemma.

So, let's unpack what it means to be a natural transformation. Over SET, a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$ is:

- An assignment to each $X \in \mathcal{C}$ an arrow $F(X) \to G(X)$, i.e. an element $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$.
- This assignment is subject to some conditions: in particular, given $\phi : Y \to Z$, we demand that

$$\begin{array}{ccc} F(Y) & \xrightarrow{\eta_Y} & G(Y) \\ {\scriptstyle F\phi}\downarrow & \# & \downarrow{\scriptstyle G\phi} \\ F(Z) & \xrightarrow{\eta_Z} & G(Z) \end{array}$$

Which is to say that $\mathrm{Nat}_{\mathcal{C}\mathcal{D}}(F, G)$ is the subset of $\prod_{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ satisfying this diagram. Restating the diagram, if $\eta \in \prod_{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$, then for every morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ we can pre- or post-compose with η: $F(Y) \xrightarrow{\eta_Y} G(Y) \xrightarrow{G\phi} G(Z)$ and $F(Y) \xrightarrow{F\phi} F(Z) \xrightarrow{\eta_Z} G(Z)$. I.e. η gives us two maps, for each $Y, Z \in \mathcal{C}$, from $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z))$. The definition of a natural transformation is exactly any such η so that these two maps are the same.

But, moving even higher, by the previous paragraph we have, for every pair $X, Y \in \mathcal{C}$, two canonical maps $\prod_{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(X), G(X)) \rightrightarrows \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(Y, Z), \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z)))$ and thus there are two maps

$$\prod_{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(X), G(X)) \rightrightarrows \prod_{Y,Z \in \mathcal{C}} \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(Y, Z), \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z)))$$

Then $\mathrm{Nat}(F, G)$ is the equalizer of these two maps (in Set, those elements that have the same image under both maps).

Well, if we can define these two maps, then we can take this as a definition of $\mathrm{Nat}(-, -)$ even in the enriched case. (Provided, of course, that we deal with the problem that if $\mathcal{C}$ is not small, then these products are not defined even in Set.)

Assuming $\mathcal{A}$ has all limits (even very large ones), we get for each Y, Z a distinguished element of

$$\mathrm{Hom}_0\left(\prod_{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(X), G(X)), \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{D}}(F(Y), F(Z)), \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z)))\right)$$

by first projecting onto the Zth component and then using the adjunction with the composition $\mathrm{Hom}_{\mathcal{D}}(F(Y), F(Z)) \otimes \mathrm{Hom}_{\mathcal{D}}(F(Z), G(Z)) \xrightarrow{\mu_{\mathcal{D}}} \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z))$. On the other hand, the functoriality of F (and the adjunction) picks a map

$$1 \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(Y, Z), \mathrm{Hom}_{\mathcal{D}}(F(Y), F(Z)))$$

and so tensoring on the right with the identity on

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{D}}(F(Y), F(Z)), \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z)))$$

and composing with the composition in $\mathcal{A}$ gives

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{D}}(F(Y), F(Z)), \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z))) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(Y, Z), \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z)))$$

But composing this with the distinguished element above gives us a map

$$\prod_{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(X), G(X)) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(Y, Z), \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z)))$$

and we can take the product over all Y and Z.

For the other map, we do almost exactly the same thing: we project onto the Yth component in the product, but now after using the adjunction for the first time and before composing we must braid $\mathrm{Hom}_{\mathcal{D}}(G(Y), G(Z))$ past $\mathrm{Hom}_{\mathcal{D}}(F(Y), G(Y))$. That we need a braiding is not too surprising — Yoneda's lemma is inherently contravariant, and to make sense of contravariant $\mathcal{A}$-functors requires the braiding in order to define $\mathcal{C}^{\mathrm{op}}$. Thus, we really do have two maps

$$\prod_{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(X), G(X)) \rightrightarrows \prod_{Y,Z \in \mathcal{C}} \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(Y, Z), \mathrm{Hom}_{\mathcal{D}}(F(Y), G(Z)))$$

corresponding to the left- and right- compositions with our proposed natural transformations. Then we define $\mathrm{Nat}_{\mathcal{C},\mathcal{D}}(F, G)$ to be the equalizer of these two maps.

A few remarks now are in order. If $T(-,-) : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{D}$ is contravariant in the first variable and covariant in the second, then we can read off a notion of a "natural family of maps" $\lambda_X \in \mathrm{Hom}_{\mathcal{D}_0}(L, T(X, X))$ as being those maps that make this square commute:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(Y, Z) & \xrightarrow{T(\mathrm{id}|_Y, -)} & \mathrm{Hom}_{\mathcal{D}}(T(Y, Y), T(Y, Z)) \\
{\scriptstyle T(\mathrm{id}|_Z, -)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(\lambda_Y, \mathrm{id})} \\
\mathrm{Hom}_{\mathcal{D}}(T(Z, Z), T(Y, Z)) & \xrightarrow{\mathrm{Hom}(\lambda_Z, \mathrm{id})} & \mathrm{Hom}_{\mathcal{D}}(L, T(Y, Z))
\end{array}$$

Then our definition of $\mathrm{Nat}_{\mathcal{C},\mathcal{D}}(F, G)$ is as the universal natural family for $T(-,-) = \mathrm{Hom}_{\mathcal{D}}(F(-), G(-))$. Any construction using identities, Homs, and adjunctions is necessarily natural; e.g. evaluation.

More comments: we have assumed a braiding in $\mathcal{A}$. Otherwise, the adjunction is just not strong enough to reproduce everything we want. But a much stronger assumption is that we have assumed $\mathcal{A}$ to contain all limits, so that it has products and equalizers. Really

I should say "all small limits", but we have also implicitly passed to a universe large enough that $\mathcal{C}$ is small. Given our experience with SET, neither of these rather strong assumptions should really surprise us.

There are two standard statements of Yoneda's lemma:

1. Given an $\mathcal{A}$-functor $F : \mathcal{C} \to \mathcal{A}$, we have an isomorphism

$$F(W) \cong \mathrm{Nat}_{\mathcal{C},\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(W, -), F)$$

2. The functor $\mathcal{C} \to \mathrm{Fun}_{\mathcal{A}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ given by $X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, -)$ is fully faithful. I.e.

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(Y, -), \mathrm{Hom}_{\mathcal{C}}(Z, -))$$

The second statement explicitly refers to the braiding in $\mathcal{A}$ in order to define $\mathcal{C}^{\mathrm{op}}$; constructing the isomorphism in the first statement also requires the braiding. So I will from now on assume that $\mathcal{A}$ is a braided monoidal category. As with SET, the second statement follows form the first by letting $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ be the functor $\mathrm{Hom}_{\mathcal{C}}(-, Z)$. Of course, the functor in the second statement is not generally an embedding (injection) of objects; for example, if $\mathcal{C} = \mathrm{SET}$ and $\mathcal{A}$ is the category of cardinalities and functions.

Thus, we have only to prove the first statement. Since we have defined $\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(W, -), F)$ as a limit — i.e. in terms of its universal property — we have only to check that $F(W)$ satisfies the universal property.

1. We need

$$\begin{aligned} F(W) &\to \prod_{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(W, X), F(X)) \\ &\rightrightarrows \prod_{Y,Z \in \mathcal{C}} \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(Y, Z), \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(W, Y), F(Z))) \end{aligned}$$

   The first arrow is straightforward. We describe it for each $X \in \mathcal{C}$. By the adjunction, we just need an arrow $\mathrm{Hom}_{\mathcal{C}}(W, X) \otimes F(W) \to F(X)$. By using the functoriality of F to get $\mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{A}}(F(W), F(X))$, braiding the $F(W)$ past, and then evaluating, we get the desired morphism. In fact, every step we did was natural in X, so we could be done with this direction: by the universal property, we have $F(W) \to \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(W, -), F)$.

   In more detail, when we defined the double arrows, we started by projecting onto the Y or Z components. Thus, we're reduced to understanding two maps

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(W, Z), F(Z)) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(Y, Z), \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(W, Y), F(Z)))$$

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(W, Y), F(Y)) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(Y, Z), \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(W, Y), F(Z)))$$

Unpacking the definitions reveals that these are very simple: In either case the map

$$F(W) \to \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(Y,Z), \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(W,Y), F(Z)))$$

is given (using the adjunction) by the map

$$\mathrm{Hom}_{\mathcal{C}}(W,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,Z) \otimes F(W) \to F(Z)$$

by braiding $F(W)$ past the Homs, and either composing the Homs and then using the functoriality of F and evaluating, or by first using functoriality and then composing and evaluating.

Using Set-based language, diagram-chasing shows that $F(W)$ is (isomorphic to) a subset of $\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(W,-), F)$.

2. On the other hand, if we have any other $\lambda : L \to \prod_{X \in \mathcal{C}} \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(W,X), F(X))$ equilizing the double arrows, we need to find a map $L \to F(W)$. Here rather than continuing to diagram-chase, I'll be rather glib. The maps

$$\lambda_X : L \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(W,X), F(X))$$

must be natural in X, and hence so is the map formed with the adjunction and the braiding

$$\mathrm{Hom}_{\mathcal{C}}(W,X) \to \mathrm{Hom}_{\mathcal{A}}(L, F(X))$$

but, being natural, this map must factor through

$$\mathrm{Hom}_{\mathcal{C}}(W,X) \to \mathrm{Hom}_{\mathcal{A}}(F(W), F(X)) \to \mathrm{Hom}_{\mathcal{A}}(L, F(X))$$

and hitting the second arrow with $\mathrm{Hom}_0(1,-)$ and using the classical Yoneda gives the desired map $L \to F(W)$.

All together, we have the Yoneda's lemma as desired: we have an isomorphism as $\mathcal{A}$-objects between $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Nat}(\mathrm{Hom}(Y,-), \mathrm{Hom}(X,-))$. To get this, we had to assume rather a lot about $\mathcal{A}$. We assumed it had a braiding, in order to define contravariance — we did not, however, need this braiding to be symmetrical. More troublingly, we assumed that $\mathcal{A}$ was huge, and had hugely many limits. Of course, to accommodate this, we really out to first increase our universe of sets until $\mathcal{A}$ is small, and then use the standard Yoneda to pass to the functor category $\mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Set})$. Provided that we can extend the monoidal structure, then this is a sufficient category, as all the limits we need exist as functors.

In any case, most of the time the categories any of us will ever need to work with are enriched over Set (as well as, possibly, other categories), or over some other extremely well-behaved category (e.g. a topos). So we almost never need this generalized Yoneda's lemma to study the objects of our category. Instead, this Yoneda's lemma lets us study the (inner) Hom objects: we might want to know, for instance, what all the supermorphisms are going into a supervector space.