Details at `arXiv:1307.5812`. Everything is dg over $\mathbb{R}$.

1. Motivation from algebraic topology

Let M be a d-dim oriented smooth manifold. Its de Rham homology $\mathrm{H}_\bullet(M)$ is a *graded commutative shifted Frobenius algebra* (*open*, i.e. nonunital, if M not compact). I.e. $\mathrm{H}_\bullet(M)$ is dg com coalg, and $\mathrm{H}_\bullet(M)[-d]$ is dg com alg, and these are compatible.

Question: Can we lift this structure to the chain level?

First try: Take "chains" to be $\mathrm{C}_\bullet(M) = \Omega^{d-\bullet}_{\mathrm{cpt}}(M)$, compactly supported de Rham forms. This has strict (shifted) com algebra structure. But no strict comult $\mathrm{C}_\bullet(M) \to \mathrm{C}_\bullet(M)^{\otimes 2} = \Omega^{d-\bullet}_{\mathrm{cpt}}(M^2)$ (projective $\otimes$), since would need distributions on diagonal $M \hookrightarrow M^2$. There is homotopy-coalg structure, but a priori unclear how coherent are the homotopies for Frobenius axiom.

Second try: Take "chains" to be $\mathrm{C}_\bullet(M) = \Omega^{d-\bullet}_{\mathrm{cpt,dist}}(M)$, comp. supp. *distributional* de Rham forms. Now have strict comultiplication, but problems with multipliation.

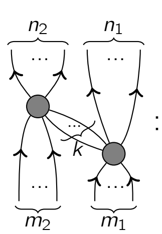
Abstract nonsense try: Take any model of chains, and choose qiso $\mathrm{C}_\bullet(M) \simeq \mathrm{H}_\bullet(M)$. Use some version of homotopy transfer theory. **Why fails?** If you did this with just (co)mult, you would never see the Massey (co)products.

2. Precisifying the problem

Defn: *Associative algebras* have compositions for each arrangement of beads on a string. Similarly:

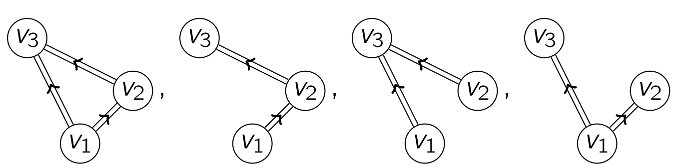
- E_d *algebras* $\leftrightarrow$ beads on $\mathbb{R}^d$
- *operads* $\leftrightarrow$ rooted trees
- *dioperads* $\leftrightarrow$ directed trees
- *properads* $\leftrightarrow$ connected acyclic directed graphs
- *props* $\leftrightarrow$ acyclic directed graphs

E.g. a *properad* P consists of $\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n$-modules $P(m, n)$ of "m-to-n operations" and *binary compositions*



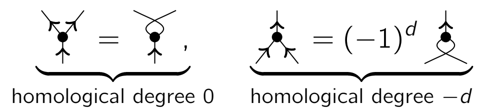
$$: \; P(m_1, k+n_1) \otimes_{\mathbb{S}_k} P(m_2+k, n_2) \to P(m_1+m_2, n_1+n_2)$$

for $k \geq 1$, satisfying associative axioms for diagrams like:

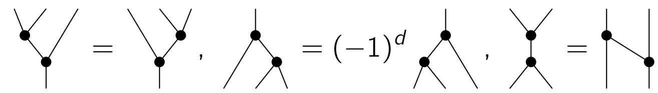


E.g.: V a chain complex. $\mathrm{End}(V)(m,n) = \hom(V^{\otimes m}, V^{\otimes n})$ defines a dioperad/properad/prop. An *action* of P on V (equiv, V is a P-*algebra*) is a morphism $P \to \mathrm{End}(V)$.

Defn: Dioperad/properad/prop Frob_d of *open d-shifted commutative Frobenius algebras* has generators:



and relations:



Thm (Vallette et al): {dioperads}, {properads}, . . . , are model categories with fibration=surjection and acyclic=qiso.

Defn: A *homotopy action* of P is an action of any cofibrant replacement $\mathrm{h}\,P$ (choice irrelevant up to homotopy).

Warning: Free : {properads} $\to$ {props} is exact, but Free : {dioperads} $\to$ {props} is not exact. So propic and properadic notions of "homotopy P-algebra" are the same, but dioperadic notion is generally different.

Question redux: Choose chain model $\mathrm{C}_\bullet(M)$. Does $\mathrm{h}\,\mathrm{Frob}_d$ act on $\mathrm{C}_\bullet(M)$ inducing Frob_d action on $\mathrm{H}_\bullet(M)$?

Avoiding abstract nonsense failure: Within $\mathrm{End}(\mathrm{C}_\bullet(M))$ is subproperad (not subprop) $\mathrm{QLoc}(M)$ of operations that "expand support only a finite amount." (In detail: for any complete metric on M, consider maps $\Omega^{d-\bullet}_{\mathrm{cpt}}(M)^{\otimes m} \to \Omega^{d-\bullet}_{\mathrm{cpt}}(M)^{\otimes n}$ with integral kernel supported in any finite-radius nbhd of diagonal M in M^{m+n}.)

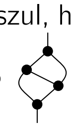
Question redux redux: $\mathrm{h}\,\mathrm{Frob}_d \to \mathrm{QLoc}(M)$?

3. Positive and negative results

Thm: With *dioperadic* interpretation, there is canonical contractible space of maps $\mathrm{h}\,\mathrm{Frob}_d \to \mathrm{QLoc}(M)$ inducing Frob_d action on $\mathrm{H}_\bullet(M)$.

Proof: (co)bar construction $\Rightarrow$ explicit presentation of $\mathrm{h}\,\mathrm{Frob}_d$. Build action inductively; at each step, look at obstructions. Calculate $\mathrm{H}_\bullet(\mathrm{QLoc}(M))$; calculate degrees of obstructions; see they must vanish.

Thm: With *properadic* interpretation, $M = \mathbb{R}$ fails.

Proof: Frob_1 is Koszul, hence get small model of $\mathrm{h}\,\mathrm{Frob}_1$. Obstruction dual to  is $-\frac{1}{12}$, which is not exact. Details at `arXiv:1308.3423`.

4. Motivation from field theory

Defn: *Classical Field Theory* = the study of those PDE determined by "least action" variational principles = geometry of critical loci in Maps(spacetime, target).

(N.B. target is usually a stack; these days derived, too.)

Defn: *QFT* = computing $\int(\text{observable})\exp\left(\frac{i}{\hbar}(\text{action})\right)$, with domain of integration Maps(spacetime, target).

Classical BV formalism: Derived critical locus of any function has symplectic form of hom degree $+1$, i.e. Poisson bracket of deg -1 (conventions: $\deg(\partial) = -1$).

BV say: *Any* dg manifold with deg-(-1) Poisson bracket should be considered as a critical locus.

Quantum BV formalism: Twisted de Rham complex for oscillating measure $\exp\left(\frac{i}{\hbar}(\text{action})\right)$ is graded com alg, with $\hbar$-dependent second-order diff. op. Δ s.t. (i) Δ is differential, (ii) $\Delta(1) = 0$, (iii) $\Delta|_{\hbar=0}$ is derivation.

BV say: *Any* graded manifold with such Δ should be considered as an oscillating integral problem.

Historical aside: Batalin–Vilkovisky were physicists, working only with $\mathbb{Z}/2$ ("super") gradings. What's called a "BV algebra" in mathematics is not what B–V discovered. It is (almost) the same with $\mathbb{Z}/2$ gradings, but different with $\mathbb{Z}$ gradings. Costello–Gwilliam name what B–V used "BD algebra," after Beilinson–Drinfeld, who used correct gradings in book on CFT.

Polemical aside: Actual derived critical loci / twisted de Rham complexes are always *cotangent bundles*. Why not work with those? Because of dualities/symmetries/gauge equivalence. Usual BV formalism keeps requirement that bracket be *symplectic*, i.e. nondegenerate.

But *symplectic is wrong*. Locally, Poisson = symplectic with parameters, and we know should study geometry in families. Globally, can have rich dualities/etc., so "families of symplectics" isn't good enough: need Poisson.

Defn: *Semistrict homotopy Pois$_d$* structure on graded algebra A is system of multiderivations making $A[1-d]$ into L_∞ alg. "Semistrict" = don't weaken Leibniz.

s.h.BD structure on graded algebra A is differential Δ on $A[\![\hbar]\!]$ such that $\frac{\partial^n}{\partial\hbar^n}|_{\hbar=0}\Delta$ is $(n+1)$th order diff. op.

Exercise: Princ. symbols of $\frac{\partial^n}{\partial\hbar^n}|_{\hbar=0}\Delta$ give s.h.Pois$_0$ str.

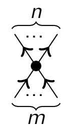
Finding Δ for prescribed s.h.Pois$_0$ str is *quantization*.

Challenge: Find interesting s.h.Pois$_0$/s.h.BD structures on mapping spaces. Interpret as classical/quantum FT.

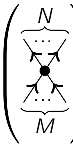
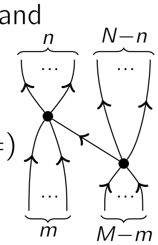
5. Connection to dioperads and properads

Focus on "infinitesimal manifolds" of type Spec $\widehat{\text{Sym}}(V)$.

Exercise: A s.h.Pois$_d$ structure on $\widehat{\text{Sym}}(V)$ is a system

 $: V^{\otimes m} \to V^{\otimes n}$ of hom degree $d(m-1)-1$

satisfying (signed) symmetry rules and

$\partial_V\Big($  $\Big) = \sum_{m,n,M-m,N-n\geq 1} (\#)$  $\quad (*)$

Coeffs $(\#)$ depend on conventions. Average over permutations of input/output strands, with signs when d = odd.

Defn: The *bar dual* $\mathbb{D}P$ of P is freely generated by $P^*[-1]$ with differential dual to $\sum$(binary compositions) : $P^{\otimes 2} \to P$ (extend as derivation; associativity $\Leftrightarrow \partial^2 = 0$.)

E.g.: Equation $(*)$ says V is alg for *dioperadic* $\mathbb{D}\,\text{Frob}_d$, and also for *properadic* $\mathbb{D}\,\text{invFrob}_d = \mathbb{D}\Big(\text{Frob}_d\,/\,(\text{⦵})\Big)$.

Exercise: s.h.BD str $\leftrightarrow$ *properadic* $\mathbb{D}\,\text{Frob}_0$ alg.

Abstract nonesense: There are canonical "sum-over-diagrams" maps $\mathbb{D}\,\text{Frob}_0 \to \text{h}\,P \otimes \mathbb{D}P$ for any P.

Application: Suppose target = Spec $\widehat{\text{Sym}}(V)$ is s.h.Pois$_d$, and M is d-dim oriented. Then $\text{Maps}\big(M_{\text{dR}}, \text{Spec}\,\widehat{\text{Sym}}(V)\big)$ = derived space of loc. constant maps $M \to \text{Spec}\,\widehat{\text{Sym}}(V)$ = $\text{Spec}\,\widehat{\text{Sym}}\big(\text{C}_\bullet(M) \otimes V\big)$ is s.h.Pois$_0$, using canonical quasilocal dioperadic h Frob$_d$ structure on $\text{C}_\bullet(M)$.

This is the *Poisson AKSZ construction*. It generalizes Alexandrov–Kontsevich–Schwarz–Zaboronsky's version for symplectic target.

6. On quantization

Suppose $\text{C}_\bullet(\mathbb{R}^d)$ has quasilocal h invFrob$_d$ action. Then get s.h.BD structure on $\text{Spec}\,\widehat{\text{Sym}}\big(\text{C}_\bullet(\mathbb{R}^d) \otimes V\big)$ for V s.h.Pois$_d$. Method of Feynman diagrams (= homological perturbation lemma = spectral sequences) applies, and gives notions of "insertion of observables," "expectation value," and "n-point function."

Thm modulo checking some details: Large-volume limit of n-point functions give $\widehat{\text{Sym}}(V)[\![\hbar]\!]$ an E_d algebra structure; thus quasilocal h invFrob$_d$ actions on $\text{C}_\bullet(\mathbb{R}^d)$ determine universal $\text{Pois}_d \to E_d$ quantization/formality.