Details are at `arXiv:1307.5812`.

For simplicity, this talk is over $\mathbb{R}$. $\exists$ a $\mathbb{Q}$ version.

1. Punchline of the talk

Conj: For $d \geq 2$, formality of the operad E_d is equivalent to formality of the properad $\mathrm{QLoc}(\mathbb{R}^d)$ satisfying:

$\mathrm{QLoc}(\mathbb{R}^d)(m,n)[-dn] = \{$cochains on $\mathbb{R}^{d(m+n)}$ with support in a finite-radius neighborhood of $\mathrm{diag}(\mathbb{R}^d) \hookrightarrow \mathbb{R}^{d(m+n)}\} \subseteq \Omega^{-\bullet}(\mathbb{R}^{d(m+n)})$

(Technical convenience: $\mathrm{QLoc}(m,n) = 0$ if $mn = 0$.)

Defn: An *associative algebra* has compositions for each way to put beads on a directed line. An *operad* has compositions for rooted trees. A *prop* has compositions for directed acyclic graphs. A *properad* has compositions for connected directed acyclic graphs.

I.e. a properad P has: • chain complexes $P(m,n)$ of "m-to-n operations" for each $m,n \in \mathbb{N}$; • actions of $\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n$; • binary operations for connected graphs with two vertices and no directed cycles; • associativity laws.

E.g.: V a chain complex. $\mathrm{End}(V)(m,n) = \hom(V^{\otimes m}, V^{\otimes n})$ defines a properad. An *action* of P on V (equivalently, V is a P*-algebra*) is a homomorphism $P \to \mathrm{End}(V)$.

E.g.: Let $\mathrm{Chains}_\bullet(\mathbb{R}^d) = \Omega^{d-\bullet}_{\mathrm{compact}}(\mathbb{R}^d)$. Then $\mathrm{QLoc}(\mathbb{R}^d)$ acts on $\mathrm{Chains}_\bullet(\mathbb{R}^d))$ and $\Omega^{d-\bullet}(\mathbb{R}^d)$. Image consists of *quasilocal* operations.

Fact: Properads form a model category with fibrations = surjections and weak equivalences = quasiisomorphisms.

(*Fibrant* means "relatively easy to map into" and *cofibrant* means "relatively easy to map out of." "Model category" implies many things, including: every P has a *cofibrant replacement* $\mathrm{h}\,P \overset{\sim}{\to} P$.)

Defn: The *space* of maps $P \to Q$ is the simplicial set whose k-simplices are maps $P \to Q \otimes \Omega_{\mathrm{PL}}(\Delta^k)$, Sullivan's polynomial forms on the k-simplex.

Defn: A *homotopy action of* P *on* V is a homomorphism $\mathrm{h}\,P \to V$. The space of such things doesn't depend on choice of $\mathrm{h}\,P$.

Defn: P is *formal* if cofibrant replacements of P and $\mathrm{H}_\bullet(P)$ are homotopy equivalent.

E.g.: $\mathrm{H}_\bullet\big(\mathrm{QLoc}(\mathbb{R}^d)(m,n)\big) = \mathbb{R}[d(1-m)]$. The nontrivial class is represented by *Thom forms*.

Algebras for $\mathrm{H}_\bullet(\mathrm{QLoc}(\mathbb{R}^d)) = \mathrm{invFrob}_d$ are *d-shifted open commutative involutive Frobenius algebras*, i.e. algebra V has noncounital cocommutative comultiplication and $V[-d]$ has nonunital commutative multiplication, satisfying a Frobenius axiom, and such that:
involutivity: (mult) $\circ$ (comult) = 0.

So formality is saying: $\mathrm{Chains}_\bullet(\mathbb{R}^d)$ is an involutive open Frobenius algebra *up to coherent homotopy*, such that all operations only move chains by finite amounts.

Remark: $\mathrm{H}_\bullet$(closed manifold) is a Frobenius algebra. For open manifolds, need interplay of $\mathrm{H}_\bullet$ and $\mathrm{H}^\bullet$. Still should expect a chain-level homotopy-Frobenius structure.

2. Technical tools for working with properads

Defn: If P is a properad satisfying mild finite-dimensionality conditions, its *bar dual* $\mathbb{D}P$ is freely generated by $P^*[-1]$, with an extra differential defined on generators to be dual to the sum of all binary compositions in P.

Fact: Under mild conditions, $\mathbb{D}P$ is cofibrant and $\mathbb{D}\mathbb{D}P \to P$ is a cofibrant replacement.

Defn: A properad P is *quadratic* if it is presented with generators T and only quadratic relations $R \subseteq T^{\otimes 2}$. The *quadratic dual* P^{i} is generated by $T^*[-1]$ with relations $R^\perp[-2]$. There is always a fibration $\mathbb{D}P \to P^{\mathrm{i}}$. P is *Koszul* if it is acyclic.

Fact: $\mathrm{invFrob}_d$ is Koszul. Its quadratic dual is LB_d — Lie bialgebras with cobracket of degree -1 and bracket of degree $d-1$. So $\mathbb{D}\,\mathrm{LB}_d \to \mathrm{invFrob}_d$ is a cofibrant replacement, much smaller than $\mathbb{D}\mathbb{D}\,\mathrm{invFrob}_d$.

Defn: Frob_0 controls regular (open) commutative Frobenius algebras, i.e. $\mathrm{Frob}_0(m,n) = \mathbb{R}$ if $mn \neq 0$.

Warning: Frob_0 is not known to be Koszul. (Its quadratic dual controls involutive Lie bialgebras. $\mathbb{D}\,\mathrm{Frob}_0$ has a generator for each connected surface with incoming and outgoing boundary.)

Fact: Under mild conditions, $\exists$ canonical homomorphism $\mathbb{D}\,\mathrm{Frob}_0 \to P \otimes \mathbb{D}P$ for each P.

Cor: Suppose $\mathrm{QLoc}(\mathbb{R}^d)$ is formal, so have $\mathrm{h}\,\mathrm{invFrob}_d \overset{\sim}{\to} \mathrm{QLoc}(\mathbb{R}^d)$. Given $\mathbb{D}\,\mathrm{invFrob}_d$-algebra V, get $\mathbb{D}\,\mathrm{Frob}_0$ action on $\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V$, with quasilocality condition.

3. Relation to geometry

Recall: Grothendieck explained how to define purely in terms of commutative algebra words like "derivation" and "differential operator." These work in dg setting. I will only use affine spaces.

Defn: A *Pois_d algebra* is a commutative algebra A with a biderivation making $A[1-d]$ into a Lie algebra. A *semistrict homotopy* Pois_d *algebra* has a system of k-fold multiderivations making $A[1-d]$ into an L_∞ algebra. "Semistrict" = don't weaken Leibniz rule.

E.g.: Poisson = Pois_1.

Lemma: A s.h.Pois_d structure on completed symmetric algebra $\widehat{\mathrm{Sym}}(V)$ is equiv to a $\mathbb{D}\,\mathrm{invFrob}_d = \mathrm{h\,LB}_d$ structure on V. (Flat, and vanishing at $0 \in V^* = \mathrm{spec}\,\widehat{\mathrm{Sym}}(V)$.)

Philosophy: $\mathrm{spec}\,\widehat{\mathrm{Sym}}(V)$ is an *infinitesimal manifold*. V are the *linear functions* for a given coordinate system. Properads control differentio-geometric structures on infinitesimal manifolds.

Lemma: A $\mathbb{D}\,\mathrm{Frob}_0$ structure on V is equivalent to an E_0 structure ∂ on $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$ such that ∂ is an nth-order diffifferential operator mod $\hbar^n$.

Defn: Such an E_0 structure on $A[\![\hbar]\!]$ makes A into a *semistrict homotopy Beilinson–Drinfeld* algebra.

Historical remark: s.h.BD structures are to BD structures as L_∞ algebras are to dglas. BD structures appear in the derived geometry of oscillating integrals, as discovered by Batalin–Vilkovisky. But Getzler defined "BV algebra" in mathematics to mean a different thing (same if you only have $\mathbb{Z}/2$ gradings) related to E_2 and Deligne conjecture. B–D were first to use the BD operad as such, and Costello–Gwilliam suggest naming it after them.

Exercise: Principal symbol of mod $\hbar^n$ part of ∂ is the nth term in a Pois_0 structure on A. This corresponds to inclusion $\mathbb{D}\,\mathrm{invFrob}_0 \hookrightarrow \mathbb{D}\,\mathrm{Frob}_0$, dual to quotient map by ideal imposing involutivity.

4. Relation to field theory

Suppose QLoc is formal, and choose V any $\mathrm{h\,LB}_d$ algebra. Then $\widehat{\mathrm{Sym}}\big(\mathrm{Chains}_\bullet(\mathbb{R}^d)\otimes V\big)[\![\hbar]\!]$ is E_0, with quasilocality condition. It is "the observables for a field theory":

$\mathrm{Chains}_\bullet(\mathbb{R}^d) \simeq \mathbb{R}$, of course, with $\leftarrow$ map depending on choice of bump function. Thus $\widehat{\mathrm{Sym}}\big(\mathrm{Chains}_\bullet(\mathbb{R}^d)\otimes V\big) \simeq \widehat{\mathrm{Sym}}(V)$. The *homological perturbation lemma* says that homotopy equivalences deform when you perturb the differential. (For oscillating integrals, HPL $\Rightarrow$ Feynman diagrams.) So there is deformed differential on $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$ making it homotopic to $\widehat{\mathrm{Sym}}\big(\mathrm{Chains}_\bullet(\mathbb{R}^d)\otimes V\big)[\![\hbar]\!]$. The *insertion* map (still) depends on a choice of bump function. The other direction is *expectation value*.

Choose bump functions near $\vec{z}_1,\dots,\vec{z}_n \in \mathbb{R}^d$. Insert $f_1,\dots,f_n \in \widehat{\mathrm{Sym}}(V)$ at those "points." Multiply the outputs in $\widehat{\mathrm{Sym}}\big(\mathrm{Chains}_\bullet(\mathbb{R}^d)\otimes V\big)[\![\hbar]\!]$. Take expectation values. This is the *n-point function*.

Theorem (modulo details — I've checked everything when $d = 1$): The n-point function depends, of course, on (the bumps near) $\vec{z}_1,\dots,\vec{z}_n$. But if all bumps have pairwise disjoint closed support, then "large volume limit" ($z_i \mapsto r z_i$, take $r \to \infty$) of n-point function converges in power-series topology. It is an n-ary multiplication, defining an E_d structure on $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$.

Cor (mod details): QLoc formailty $\Rightarrow$ E_d formality.

Proof: When $d \geq 2$, formality of E_d is equiv to having a universal quantization $\mathrm{Pois}_d \rightsquigarrow E_d$. Any Pois_d algebra, in characteristic 0, has a resolution by a polynomial algebra. Universal quantization of infinitesimal Pois_d manifolds takes polynomials to polynomials.

How to prove the converse: Let V be the universal $\mathrm{h\,LB}_d$ algebra (i.e. the generating object of the sym mon cat defined by $\mathrm{h\,LB}_d$). Universal E_d quantization gives a factorization algebra on $\mathbb{R}^d$ that assigns something homotopic to $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$ to any contractible open. In principal, we should be able to choose it to assign precisely $\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(-)\otimes V)[\![\hbar]\!]$ to any open. Unpacking the differential in terms of Feynman diagrams gives some universal operations on $\mathrm{Chains}_\bullet(-)$. These should give the formality of QLoc.

5. The Poisson AKSZ construction

Alexandrov, Kontsevich, Schwartz, and Zaboronsky, while studying Chern–Simons theory in the Batalin–Vilkovisky framework, realized that if M is an oriented closed d'-dimensional manifold and X is a *symplectic* Pois_d manifold (meaning the bracket is an isomorphism from cotangent to tangent bundles), $\underline{\mathrm{Maps}}(\mathrm{T}[1]M, X) =$ the derived space of locally constant maps from M to X is a symplectic $\mathrm{Pois}_{d-d'}$ manifold.

Suppose $X = \mathrm{spec}\,\widehat{\mathrm{Sym}}(V)$. Then $\underline{\mathrm{Maps}}(\mathrm{T}[1]M, X) = \mathrm{spec}\,\widehat{\mathrm{Sym}}\big(\mathrm{Chains}_\bullet(M)\otimes V\big)$.

A *dioperad* is like an operad or properad, but with multiplications only for directed trees. In *dioperads*, $\mathrm{LB}_d^{\mathrm{i}} = \mathrm{Frob}_d$, and they are Koszul.

Theorem: If M is oriented of dimension d', there is a canonical (i.e. contractible space of) *dioperadic* $\mathrm{h\,Frob}_{d'}$ structures on $\mathrm{Chains}_\bullet(M)$, satisfying quasilocality.

Cor: Any s.h.Pois_d structure on $\mathrm{spec}\,\widehat{\mathrm{Sym}}(V)$ gives s.h.$\mathrm{Pois}_{d-d'}$ structure to $\mathrm{spec}\,\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(M)\otimes V)$.

Defn: This is the *Poisson AKSZ construction*. It gives the AKSZ construction in the symplectic case. Extending to properads is *path integral quantization*.