Details at `arXiv:1307.5812`. Everything is dg. char=0.

0. Punchline of talk

Defn: An operation $f : \mathrm{Chains}_\bullet(\mathbb{R}^d)^{\otimes m} \to \mathrm{Chains}_\bullet(\mathbb{R}^d)^{\otimes n}$ is *quasilocal* if $\exists \ell \in \mathbb{R}$ s.t. $f(a_1 \otimes \cdots \otimes a_m)$ is in radius-ℓ nbhd of support(a_i) $\forall i$. These form properad $\mathrm{QLoc}(\mathbb{R}^d)$. (Technical convenience: force $m, n \neq 0$.)

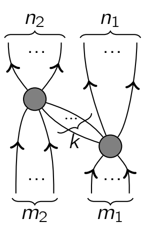
Conj: For $d \geq 2$, space of formality morphisms of properad $\mathrm{QLoc}(\mathbb{R}^d)$ is canonically homotopy equivalent to space of formality morphisms of operad E_d.

1. Some definitions

Defn: *Associative algebras* have compositions for each arrangement of beads on a string. Similarly:

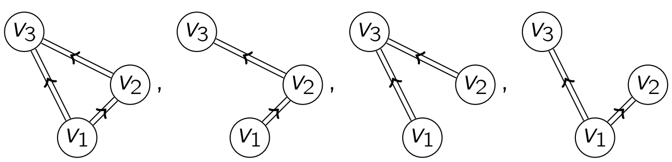
- *E_d algebras* $\leftrightarrow$ beads on $\mathbb{R}^d$
- *operads* $\leftrightarrow$ rooted trees
- *dioperads* $\leftrightarrow$ directed trees
- *props* $\leftrightarrow$ acyclic directed graphs
- *properads* $\leftrightarrow$ connected acyclic directed graphs

I.e. a *properad* P consists of $\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n$-modules $P(m,n)$ of "m-to-n operations" and *binary compositions*



$$: P(m_1, k+n_1) \otimes P(m_2+k, n_2) \to P(m_1+m_2, n_1+n_2)$$

for $k \geq 1$, satisfying associative axioms for diagrams like:



E.g.: V a chain complex. $\mathrm{End}(V)(m,n) = \hom(V^{\otimes m}, V^{\otimes n})$ defines a properad. An *action* of P on V (equivalently, V is a *P-algebra*) is a homomorphism $P \to \mathrm{End}(V)$.

E.g.: $\mathrm{QLoc} \subseteq \mathrm{End}(\mathrm{Chains}_\bullet(\mathbb{R}^d))$. QLoc also acts on $\mathrm{Cochains}^{d-\bullet}(\mathbb{R}^d)$, compatibly with Chains $\hookrightarrow$ Cochains.

Thm (Vallette): Properads and props form model categories with fibration=surjection and acyclic=quasi-iso. Free : {properads} $\to$ {props} is exact.

Warning: Free : {dioperads} $\to$ {props} is not exact.

Notation: $\mathrm{h}\,P$ is any cofibrant replacement of P.

Defn: The *space* of maps $\mathrm{h}\,P \to Q$ is the simplicial set whose k-simplices are maps $\mathrm{h}\,P \to Q \otimes \Omega_{\mathrm{PL}}(\Delta^k)$. ($\Omega_{\mathrm{PL}}(\Delta^k) =$ Sullivan's polynomial forms on the k-simplex.)

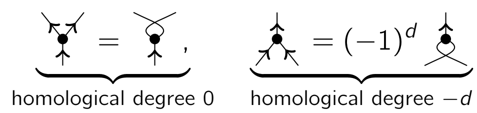
Fact: Different choices for $\mathrm{h}\,P$ give homotopy equiv spaces.

Defn: A *formality morphism* of X is a homotopy equiv between $\mathrm{h}\,X$ and $\mathrm{h}\,\mathrm{H}_\bullet(X)$, covering identity on $\mathrm{H}_\bullet$.

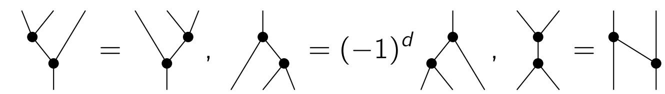
There are enough filtrations that $\mathrm{H}_\bullet \approx$ associated graded.

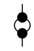
E.g.: $\mathrm{H}_\bullet(E_d) = \mathrm{Pois}_d$ ($d \geq 2$): com algs with Poisson bracket of deg $d-1$. Formality $\approx$ universal quantization.

Defn: $\mathrm{H}_\bullet(\mathrm{QLoc}(\mathbb{R}^d)) = \mathrm{invFrob}_d$ controls *open d-shifted involutive Frobenius algebras*. Generators (read up):



Associativity and Frobenius relations:



Involutivity:  $= 0$.

2. Technical tools

Defn: Under mild finiteness conditions, *bar dual* $\mathbb{D}P$ of properad P is free properad generated by $P^*[-1]$, with differential encoding binary compositions in P.

Fact: $\mathbb{D}P$ is cofibrant. $\mathbb{D}\mathbb{D}P = \mathrm{h}\,P$ works, but it's big.

Fact: Good theory of quadratic and Koszul properads.

Defn: Frob_0 controls usual open com Frob algs.

Fact: $\mathrm{invFrob}_d$ is Koszul $\forall d$. Frob_0 is unknown.

Lemma: $\forall P$, $\exists$ canonical map $\mathbb{D}\,\mathrm{Frob}_0 \to \mathbb{D}P \otimes P$.

Remark: In dioperads, involutivity doesn't make sense. Frob_d is Koszul $\forall d$. Still have $\mathbb{D}\,\mathrm{Frob}_0 \to \mathbb{D}P \otimes P$.

3. Geometry and physics

Defn: *QFT* = computing $\int(\text{observable}) \exp\left(\frac{i}{\hbar}(\text{action})\right)$.

The *Batalin–Vilkovisky formalism* identifies oscillating integrals with the following geometry:

Defn: A *Beilinson–Drinfeld manifold* X has $\hbar$-dependent differential $\Delta : \mathcal{O}(X) \to \mathcal{O}(X)$ such that: 1. $\partial(1) = 0$. 2. Δ is second-order diff op. 3. $\Delta|_{\hbar=0}$ is derivation.

Historical remark: Mathematicians' "BV structure" is related to E_2 and Deligne conjecture; it's similar, but differs by a sign in the $\mathbb{Z}$-gradings. (This is Getzler's fault.) B–D got it right (in their CFT book), so Costello–Gwilliam name the BD operad after them.

Exercise: So $(\mathcal{O}(X), \Delta|_{\hbar=0})$ is dgca. Principal symbol of $\frac{\partial}{\partial\hbar}|_{\hbar=0}\Delta$ makes $(\mathcal{O}(X)[1], \Delta|_{\hbar=0})$ into dgla, i.e. $\mathcal{O}(X)$ into Pois_0 algebra.

A polemical aside: In usual BV formalism, the Pois_0 structure is required to be *symplectic*. (Actual oscillating integrals $\leftrightarrow$ cotangent bundles.) Locally, Poisson = symplectic with parameters. Global Poisson topology comes from symmetry / duality between theories. *So restricting just to symplectic things is wrong.*

Defn:

dgla : L_∞ :: Pois_d : *semistrict homotopy* Pois_d

= system of multiderivations on $\mathcal{O}(X)$ making $\mathcal{O}(X)[1-d]$ into L_∞ alg. "semistrict" = don't weaken Leibniz.

Defn:

dgla : L_∞ :: BD : s.h.BD

= $\hbar$-dependent E_0 structure Δ on $\mathcal{O}(X)$ s.t. $\frac{\partial^n}{\partial\hbar^n}|_{\hbar=0}\Delta$ is $(n+1)$-order differential operator.

Exercise: Principal symbols turn s.h.BD into s.h.Pois_0.

Connection to properads: An *infinitesimal manifold* is $X = \mathrm{spec}\,\widehat{\mathrm{Sym}}(V)$.

s.h.Pois_d str on X = $\mathbb{D}\,\mathrm{invFrob}_d$ str on V.

Let's declare $\hbar$ to be formal variable. Then:

s.h.BD str on X = $\mathbb{D}\,\mathrm{Frob}_0$ str on V.

Remark: Above is *properadic* $\mathbb{D}$. In general,

dioperadic $\mathbb{D}\,\mathrm{Frob}_d$ = properadic $\mathbb{D}\,\mathrm{invFrob}_d$.

4. The AKSZ construction

BV Formalism: Pois_0 structures pose oscillating integral problems. How to find Pois_0 strs on spaces of "fields"?

Thm (Alexandrov–Kontsevich–Schwarz–Zaboronsky): M is closed oriented d-dim manifold. X is *symplectic* Pois_d. Then $\mathrm{Maps}(M_{\mathrm{dR}}, X)$ = derived space of locally constant maps $M \to X$ is symplectic Pois_0.

With one lie. It is symplectic. But it's ∞-dim. How to invert to Poisson structure? (And see earlier polemic.)

I have an answer when $X = \mathrm{spec}\,\widehat{\mathrm{Sym}}(V) \approx V^*$.

$\mathrm{Maps}(M_{\mathrm{dR}}, V^*) = \mathcal{O}(M_{\mathrm{dR}}) \otimes V^* = \Omega^\bullet(M) \otimes V^*$. Linear functions thereon $= (\Omega^\bullet(M) \otimes V^*)^* = \mathrm{Chains}_\bullet(M) \otimes V$.

We are given a $\mathbb{D}\,\mathrm{invFrob}_d$ structure on V. We want a $\mathbb{D}\,\mathrm{invFrob}_0$ structure on $\mathrm{Chains}_\bullet(M) \otimes V$. Or, working dioperadically, given $\mathbb{D}\,\mathrm{Frob}_d \to \mathrm{End}(V)$, find $\mathbb{D}\,\mathrm{Frob}_0 \to \mathrm{End}(\mathrm{Chains}_\bullet(M) \otimes V) = \mathrm{End}(\mathrm{Chains}_\bullet(M)) \otimes \mathrm{End}(V)$.

Thm: There is a canonical contractible space of quasilocal actions of *dioperadic* $\mathrm{h}\,\mathrm{Frob}_d = \mathbb{D}\mathbb{D}\,\mathrm{Frob}_d$ on $\mathrm{Chains}_\bullet(M)$.

Defn: Dioperadic $\mathbb{D}\,\mathrm{Frob}_0 \to \mathbb{D}\mathbb{D}\,\mathrm{Frob}_d \otimes \mathbb{D}\,\mathrm{Frob}_d \to \mathrm{End}(\mathrm{Chains}_\bullet(M)) \otimes \mathrm{End}(V)$ is the *classical Poisson AKSZ construction*.

What about quantum? Need properadic $\mathrm{h}\,\mathrm{invFrob}_d \to \mathrm{QLoc}(M)$. When M closed and $\chi(M) \neq 0$, this definitely can't happen.

5. Relation to conjecture

Formality of $\mathrm{QLoc}(\mathbb{R}^d)$ = quasilocal $\mathrm{h}\,\mathrm{invFrob}_d$ action on $\mathrm{Chains}_\bullet(\mathbb{R}^d)$. This gives a way to turn s.h.Pois_d strs on $\widehat{\mathrm{Sym}}(V)$ into s.h.BD strs on $\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V)$, i.e. E_0 strs on $\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V)[\![\hbar]\!]$.

Defn: $\mathrm{Chains}_\bullet(\mathbb{R}^d) \simeq \mathbb{R} \Rightarrow \widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V) \simeq \widehat{\mathrm{Sym}}(V)$. Homological perturbation lemma (= Feynman diagrams) $\Rightarrow$ still true after deforming.

Map $\mathbb{R} \xrightarrow{\sim} \mathrm{Chains}_\bullet(\mathbb{R}^d)$ required choosing $\vec{z} \in \mathbb{R}^d$ (or a bump function). Still true after deforming. New map $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!] \to \widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V)[\![\hbar]\!]$ is the *insertion of an observable at* $\vec{z}$. Deformed map $\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V)[\![\hbar]\!] \to \widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$ is *expectation value*.

Choose $\vec{z}_1, \dots, \vec{z}_n \in \mathbb{R}^d$. Insert $f_1, \dots, f_n \in \widehat{\mathrm{Sym}}(V)$, multiply in $\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(\mathbb{R}^d) \otimes V)[\![\hbar]\!] \to \widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$, and take expectation values. This is the *n-point function*.

Theorem (modulo details — I've checked everything when $d = 1$): The n-point function depends, of course, on (bumps near) $\vec{z}_1, \dots, \vec{z}_n$. But if all bumps have pairwise disjoint closed support, then "large volume limit" ($z_i \mapsto rz_i$, take $r \to \infty$) of n-point function converges in power-series topology. It is an n-ary multiplication, part of an E_d structure on $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$.

Cor (mod details): QLoc formailty $\Rightarrow$ universal (wheel-free) E_d quantization for infinitesimal manifolds $\Rightarrow$ universal E_d quantization $\Leftrightarrow$ E_d formality when $d \geq 2$.

Idea for the converse: E_d quantization $\Rightarrow$ quantization of factorization algebra $\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(-) \otimes V)$ over $\mathbb{R}^d$, for V the universal $\mathbb{D}\,\mathrm{invFrob}_d$-algebra. Unpack the Feynman diagrams: get universal operations on $\mathrm{Chains}_\bullet(\mathbb{R}^d)$. Hopefully, these are a quasilocal action of $\mathrm{h}\,\mathrm{invFrob}_d$.

Warning: When $d = 1$, $\mathrm{QLoc}(\mathbb{R}^1)$ is not formal. See `arXiv:1308.3423`. $\nexists$ universal E_1 quantization.