Twisted $\mathcal{N}=2$ supersymmetry on $\mathbb{R}^4$

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Abstract: The goal of this talk is to explain the title.

This talk is based in its entirety on a talk I saw last week by Kevin Costello.

Translation-invariant field theory on $\mathbb{R}^4$

I won't really try to define what a "field theory" is. It should include some data: maybe some "fields," probably some "observables," etc., and these should form a $\mathbb{C}$-vector space. fields and observables and so on have "geographic" data. For example, maybe you think a "field theory" consists of "field operators" located at points.

There is an $\mathbb{R}^4$ acting on spacetime $= \mathbb{R}^4$ by translations. The field theory is "translation-invariant" if this action lifts to the space of fields, observables, operators, ..., equivariant for the "geographic" data, and such that the "physics" is invariant for this action. Similarly we can talk about "rotation-invariant" theories where we ask the "physics" to be invariant for a $\mathrm{Spin}(4)$ (in Euclidean signature) or $\mathrm{Spin}(3,1)$ (Minkowski) action.

Of course, our space of observables is a $\mathbb{C}$-vector space, so an action by $\mathbb{R}^4$ [$\mathrm{Spin}(4)$ or $\mathrm{Spin}(3,1)$] lifts to an action by $\mathbb{C}^4$ [$\mathrm{Spin}(4,\mathbb{C})$].

For the rest of the talk, for definiteness I will assume we are in Euclidean signature, but since mostly everything will be complexified, this won't really matter.

Eventually I'll assume that our "physics" is determined by some BRST/BV operator: the spaces of observables, operators, ... are ($\mathbb{C}$-linear) chain complexes.

SUSY

$\mathrm{Spin}(n)$ = double cover of $SO(n)$ always has two special representations, the positive and negative spinors. When $n=4$ these are particularly easy to define.

First, the $\mathrm{Spin}(4)$ action on $\mathbb{R}^4$: identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. Then $U(1,\mathbb{H})$ = unit quaternions acts on $\mathbb{H}$ by left multiplication and there's a commuting action by right multiplication. So we have $U(1,\mathbb{H}) \times U(1,\mathbb{H}) \to SO(4)$, with kernel $\{\pm 1\}$, and a dimension count verifies

$$\mathrm{Spin}(4) = U(1,\mathbb{H}) \times U(1,\mathbb{H}).$$

Of course, $U(1,\mathbb{H}) = \mathrm{Spin}(3) = SU(2)$. To see this, we can write $\mathbb{H} = \mathbb{C} \oplus J \cdot \mathbb{C}$ where $\mathbb{C}$ is spanned by $\{1, I\}$. Then the left action of $U(1,\mathbb{H})$ is right-$\mathbb{C}$-linear.

We set $\$_+$ to be the 2-complex-dimensional representation of $\mathrm{Spin}(4)$ in which the left $SU(2)$ acts on $\mathbb{H} = \mathbb{C}^2 = \mathbb{C} \oplus J\mathbb{C}$ by left multiplication and the right $SU(2)$ acts trivially. We set $\$_-$ to be the $\mathbb{C}^2$ on which the right $SU(2)$ acts by right multiplication and the left acts trivially. Then it's more or less clear that

$$\$_+ \otimes \$_- = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\text{-span of } \{1, J, K, JK\}$$
$$= \text{"vector representation of } Spin(4)\text{"}$$
$$= \mathbb{C}^4.$$

Defn: The <u>$\mathcal{N}=1$ supertranslation group</u> is the super Lie algebra on $\mathbb{C}^4 \oplus \pi(\$_+ \oplus \$_-)$, where "$\pi$" denotes that those parts are "odd", with bracket

$$[\,,\,] : \pi\$_+ \otimes \pi\$_- \longrightarrow \mathbb{C}^4 =$$

the canonical map of $Spin(4)$-modules, and all other brackets are 0. This is a central extension of an abelian Lie algebra:

$$\mathbb{C}^4 \longrightarrow \text{supertranslation} \longrightarrow \pi(\$_+ \oplus \$_-).$$

Eventually we will assign $\mathbb{Z}$-gradings with $\$_\pm$ in degree ± 1.

Defn: A <u>supersymmetric translation-invariant</u> field theory is a translation-invariant field theory in which the $\mathbb{R}^4$ action on operators, ..., is extended to an action by the supertranslation group.

<u>Defn</u>: The <u>super Poincaré group</u> is

$$\text{superPoincaré} = Spin(4) \ltimes \text{supertranslation}.$$

You can guess what is a supersymmetric Poincaré-invariant field theory.

<u>Defn</u>: Let W be an $\mathcal{N}$-dimensional $\mathbb{C}$-vector space. Then $\mathcal{N}=\mathcal{N}$ SUSY translation group is the central extension

$$\mathbb{C}^4 \longrightarrow (\mathcal{N}=\mathcal{N} \text{ SUSY}) \longrightarrow \Pi(W \otimes \$_+ \oplus W^* \otimes \$_-)$$

with bracket

$$[\,,\,] : W \otimes \$_+ \otimes W^* \otimes \$_- \to \mathbb{C}^4$$

given by pairing the $\$_\pm$ parts to a vector and using the canonical pairing $W \otimes W^* \to \mathbb{C}$.

We similarly have $\mathcal{N}=\mathcal{N}$ superPoincare as a semidirect product.

<u>Defn</u>: Choose $G \subseteq GL(W)$. [redacted] An [redacted] <u>$\mathcal{N}=\mathcal{N}$ SUSY field theory with R-symmetry G</u> is a field theory with compatible action by $G \ltimes$ (SUSY group).

<u>Twisting</u>

Just to choose conventions, let's use <u>cohomological</u> gradings, meaning that our "physics" consists of some sort of dg algebraic-object with "BRST/BV" differential d in degree $+1$. Recall that we graded the SUSY group by

$$\mathbb{C}^4 \oplus W \otimes \$_+ \oplus W^* \otimes \$_-$$

degree: $\quad 0 \qquad 1 \qquad -1$.

Pick $Q \in W \otimes \$_+$. Then Q acts on our system and $[d, Q] = 0$ and $[Q, Q] = 0$ — well, at least, in dg world $[Q,Q] = d(\text{something})$ with higher coherences, but I'll use notation as if $[Q,Q]=0$. Anyway, the yoga of dg algebra says that:

$d + tQ$ is a new differential on the same system,

where t is a formal variable.

Half of this statement is completely obvious: $[d+tQ, d+tQ] = [d,d] + 2t[d,Q] + t^2[Q,Q] = 0$. Why must t be a formal variable? One reason is that in fact we only have $[Q,Q] = [d, \text{something}]$ and so we actually will need to deform to $d + tQ + t^2\,\text{something} + \dots$.

But even when $[Q,Q]=0$ on the nose, we need t to be formal because we will want to transfe the deformation $d \rightsquigarrow d + tQ$ to other homotopy-equivalent systems, and for the transfer lemma to work we need some convergence.

We can do better if we have an R-symmetry. Suppose in particular that we choose a torus $\mathbb{C}^\times \hookrightarrow GL(W)$ and our system has symmetry $\mathbb{C}^\times \ltimes$ supertranslations, and suppose that Q is weight-1 for this $\mathbb{C}^\times$ action.

Claim: In this case, we can let t be an algebraic parameter in $\mathbb{C}^\times$. In particular, we can evaluate at $t=1$.

Why?. Idea: we can use the R-symmetry to rescale Q, and so any value of t is isomorphic to any other.

Mathematically: Recall that a $\mathbb{C}^\times$-equivariant sprocket over $\mathbb{C}$ is the same thing as a filtered sprocket: the fiber over $0 \in \mathbb{C}$ is the associated graded sprocket, and the generic fiber is the actual sprocket (having forgotten its grading).

If you like spectral sequences, then you know that a filtered dg sprocket gives rise to one, which implements the homological transfer. Equivalently, convergence in the homological perturbation lemma is assured by the filtration.

<u>Defn:</u> A susy field theory <u>with twisting data</u> is a susy field theory with R-symmetry $\mathbb{C}^\times \hookrightarrow GL(W)$ and a choice of $Q \in W \otimes \$_+$ which is weight-1 for the $\mathbb{C}^\times$ action. The corresponding <u>twisted theory</u> is the theory with the BRST/BV operator d replaced by $d+Q$.

<u>What happens when you twist?</u>

A basic fact of homological perturbation theory is that the dimensions of the homology groups are upper-semicontinuous for deformations of d: deforming d makes the homology shrink. A basic fact of homological physics is that the "physical" observables, operators, ... are the homology of the dg spaces of all observables. So the twisted theory is <u>simpler</u> than the untwisted one: it is a potentially computable piece of the original theory, where observables, ..., depend on less physical data.

Let's focus on $\boxed{\mathcal{N}=1}$ with R-symmetry $\mathbb{C}^\times = GL(1)$.

We choose non-zero $Q \in \$_+$. The supertranslation group deforms to a dg Lie group by picking up the differential $[Q,-]$:

$$\$_- \xrightarrow{\substack{Q\otimes \\ \| \\ [Q,-]}} \mathbb{C}^4 \qquad \$_+$$

degree: $-1 \qquad 0 \qquad 1$

So the homology is 2-dim'l in ~~g~~ degree 0 and 2-dim'l in degree 1.

This group acts on the deformed theory. In particular, the exact part $\$_- \xrightarrow{Q} \mathbb{C}^4$ must act cohomologically-trivial.

Recall that this $\mathbb{C}^4$ is the complexified group of translations. Call the image of $\$_-$ in $\mathbb{C}^4$ the span of $\bar\partial_1$ and $\bar\partial_2$. We still have complex conjugation in $\mathbb{C}^4$; call the conjugate basis vectors ∂_1 and ∂_2.

<u>Exercise</u>: $\bar\partial_1, \bar\partial_2, \partial_1, \partial_2$ form a basis of $\mathbb{C}^4$.

Now, $\mathbb{C}^4 = T_{pt}\mathbb{R}^4 \otimes \mathbb{C}$, and we've split it as $\langle \bar\partial_1, \bar\partial_2 \rangle \oplus \langle \partial_1, \partial_2 \rangle$. These are the $T^{0,1}$ and $T^{1,0}$ parts of a Kähler structure on $\mathbb{R}^4$.

So in the deformed physics, we have $\bar\partial$ operators that act (cohomologically) trivially.

Thus, <u>the twisted theory</u> <u>is a holomorphic theory on $\mathbb{C}^2$</u>. More precisely, it's the "derived" version of this: rather than being defined over $\mathcal{C}^\infty(\mathbb{R}^4)$, it exists over the

Dolbeaut complex.

Finally, turn your attention to $\boxed{\mathcal{N}=2}$.
A twisting should come from $\$_+ \otimes \mathbb{C}^2$ $\cong \hom(\mathbb{C}^2, \mathbb{C}^2)$. There are three possibilities:

(0) $Q=0$. $\rightsquigarrow$ untwisted theory.

(1) Q is a pure tensor, i.e. corresponding 2×2 matrix has rank 1.

(2) generic case: matrix has rank 2.

In the pure tensor case, our dg Lie group

$$W^* \otimes \$_- \xrightarrow{[Q,-]} \mathbb{C}^4 \qquad W \otimes \$_+$$

has cohomology

$$\$_- \qquad \mathbb{C}^2 \qquad \$_+ \oplus \$_+ .$$

In particular, we still have two nontrivial translation directions ∂_1, ∂_2, but we again are a holomorphic theory: $\bar{\partial}_1$ and $\bar{\partial}_2$ are 0 (in cohomology).

In the generic case, $[Q,-] : (\$_- \oplus \$_-) \to \mathbb{C}^4$ is an isomorphism. So <u>all</u> translations act trivially, and the theory is (framed) <u>topological</u>: the value of an observable, operator, ... <u>does not depend on its location</u>. SUSY field theories require a notion of "translation-invariance" and so (usually) don't make sense on manifolds other than $\mathbb{R}^n$. (Question: Is there a version of SUSY on other homogeneous spaces?) On the other hand, many twisted theories can be extended to complex surfaces, say, or framed manifolds.

Summary:

- Twisting makes a SUSY theory simpler.
- Generic $\mathcal{N}=1$ twisted SUSY field theory on $\mathbb{R}^4$ is holomorphic.
- Generic $\mathcal{N}=2$ twisted SUSY field theory on $\mathbb{R}^4$ is topological.