Wick's Theorem beyond the Gaussian

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Thank you to Ben for the invitation to speak here. Oh, and lest I forget, let me mention that what I'll tell you is joint work with Owen Gwilliam and joint work in progress with Shamil Shakirov. (Well, at least the work with Owen is probably known to some experts, but we never saw it written down. The work with Shamil is still in its beginning stages.)

Introduction

Before I begin with the real mathematics, I'd like to set my story in a little bit of context. I am primarily a quantum field theorist. One of the basic tenets of quantum field theory is that:

> Numbers of physical interest tend to arise as integrals

and in fact as expectation values for probability measures. The problem is that these integrals are rarely analytically defined:

> over spaces that do not support analytic definitions of integration.

The spaces that one would like to compute integrals over tend to be very infinite-dimensional, highly stacky, etc.

Indeed, the spaces may not even be analytic objects at all. The phase space of the universe is probably an algebraic variety, with dynamics controlled by algebraic differential equations. So all answers should be numbers with some algebraic properties (periods. . .).

So an ongoing project in quantum field theory is:

> Goal: The algebraization of integration.

To restrict the problem a little bit, I will describe in more detail the types of integrals that seem to arise in quantum field theory. There tends to be some naturally occurring "Lebesgue measure" dLeb in some variables, and a distinguished polynomial function s in those variables (s is the first letter of "action"), and the measure you actually want is $\mu = \exp(s)\mathrm{dLeb}$, and you want to compute integrals (or maybe just expectation values) of polynomials.

$$\langle f \rangle_\mu = \int f\, e^s \,\mathrm{dLeb}$$

Anyway, there are at least three questions you can ask about such problems:

1. **Undergraduate calculus:** Given an algebraic description of μ (formula for s), compute $\langle f \rangle_\mu$ for various f.
2. **Inverse problem:** Given some values of $\langle f \rangle_\mu$ for various f, and an a priori restriction on the "shape" of μ (e.g. the degree of s), compute μ.
3. **Predicting experiments:** Given some values of $\langle f \rangle_\mu$ and restriction on the shape of μ, compute $\langle g \rangle_\mu$ for other functions g.

Usually in mathematics classes we only discuss problems of type 1, which is why I've labeled it "undergraduate calculus". But it's the third type of problem that's most important for physics, because what we actually can measure are some values $\langle f \rangle_\mu$, and we'd like to predict more and then test them. Unfortunately, I don't have many exciting results for this third question, although I'll mention some ideas at the end of my talk. Instead, I'll spent most of my talk discussing the first question.

The ur-example, which would be great to generalize, is:

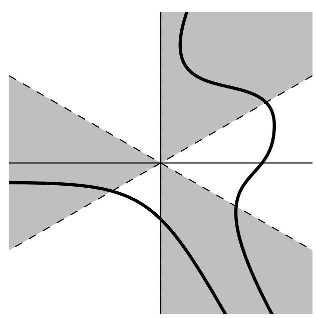
> **Wick's Theorem (Isserlis, 1918):** Suppose we are integrating over $\mathbb{R}^n$ and that s is homogeneous quadratic, and identify such functions with symmetric matrices by $s(x) = \sum_{ij} s_{ij} x_i x_j / 2$. Suppose that s has negative-definite real part (so that integrals over $\mathbb{R}^n$ converge). Then the inverse matrix exists, and I'll write $(s^{-1})_{ij}$ for its inverse matrix. Suppose further that f is a homogeneous polynomial of degree m. Then:
>
> 1. $\langle f \rangle = 0$ if m is odd. If m is even, then
>
> $$\langle f \rangle = \det(-2\pi s)^{-1/2} \frac{1}{(m/2)!} \left(\sum_{ij} \frac{(s^{-1})_{ij}}{4} \frac{\partial^2}{\partial x_i \partial x_j} \right)^{m/2} f$$
>
> 2. The data of s can be recovered as the inverse matrix to $\frac{\langle x_i x_j \rangle}{\langle 1 \rangle}$.
> 3. The value of $\langle f \rangle$ can be computed from the value of $\langle 1 \rangle$ and the values of $\langle x_i x_j \rangle$ just for those variables x_i and x_j that appear in f. Indeed, even if $s(x)$ is inhomogeneous quadratic (with invertible homogeneous quadratic part), then you can compute arbitrary $\langle f \rangle$ from the data of $\langle 1 \rangle$, $\langle x_i \rangle$, and $\langle x_i x_j \rangle$, and you only need the values for those variables appearing in f.

One thing to draw attention to in the last condition is that if means that you can let n go off to infinity: Wick's Theorem can be taken as a *definition* of "Gaussian integration" in infinite-dimensional space, and to do physics you never need to know *all* the degrees of freedom. Note also the following: We assumed that all eigenvalues of s have negative real part, and so the branch of $(\det)^{-1/2}$ is unambiguous. But actually we can choose a branch cut along the set of s with a non-negative real eigenvalue, and if we do so then the formula in 1. makes sense for, say, invertible purely-imaginary s, whence it correctly computes the corresponding conditionally-convergent integrals.

In the real world, one application of Wick's theorem is the following. You posit not that s is homogeneous quadratic, but that it is quadratic plus some very small (infinitesimal coupling constant) perturbation. Then the failure of 3. to hold exactly gives you data about the values of the perturbation. This is the world of Feynman diagrams.

The problem, contours, and Stokes' theorem

I will discuss the following situation. We choose a homogeneous complex-valued polynomial s of degree d in n variables, and consider integrals of the form $\langle f\rangle = \int f\, e^s\, \mathrm{dLeb}$ for f a polynomial. This requires choosing a contour γ of integration: a real-n-dimensional subspace of $\mathbb{C}^n$. In order for the integral to converge we would like exponential decay at the ends of γ — i.e. we ask that $s \to -\infty$ along γ. For example, when $n = 1$ and $s(x) = x^3$, here are some allowable contours:



Thus the action s really picks out a pairing:

$$(f, \gamma) \mapsto \int_\gamma f\, e^s\, \mathrm{dLeb} : \quad \{\text{polynomials}\} \times \{\text{allowed contours}\} \to \mathbb{C}$$

I'm writing dLeb, but I really mean its holomorphic extension $\mathrm{d}z_1 \cdots \mathrm{d}z_n$, which is a holomorphic n-form on $\mathbb{C}^n$.

Of course, we don't really need all contours:

> **Stokes' Theorem:** The value of $\int_\gamma f\, e^s\, \mathrm{dLeb}$ only depends on the class of γ in the relative homology group $\mathrm{H}_n(\mathbb{C}^n, \{\Re(s) < 0\})$.

When $n > 1$ (so that $\mathrm{H}_n(\mathbb{C}^n) = 0$), the long exact sequence for relative homology gives an isomorphism $\mathrm{H}_n(\mathbb{C}^n, \{\Re(s) < 0\}) = \mathrm{H}_{n-1}(\{\Re(s) < 0\})$. When $n = 1$, we have $\mathrm{H}_1(\mathbb{C}^n, \{\Re(z^d) < 0\}) = \mathrm{H}_0(\mathbb{C}, \{\Re(z^d) < 0\}) / \mathrm{H}_0(\mathbb{C})$ is $(d-1)$-dimensional. For example, the above two contours are a basis for the relevant space.

We also don't need all polynomials:

> **Stokes' Theorem:** The value of $\int_\gamma f\, e^s \, \mathrm{dLeb}$ only depends on the class of f in the quotient space $\mathbb{C}[x_1, \dots, x_n]/(\frac{\partial f}{\partial x_i} + f\, \frac{\partial s}{\partial x_i} \equiv 0)$.

All together, the important pairing is:

$$\frac{\mathbb{C}[x_1, \dots, x_n]}{(\text{image of } \frac{\partial}{\partial x_i} + \frac{\partial s}{\partial x_i} \cdot)} \otimes \mathrm{H}_n(\mathbb{C}^n, \{\Re(s) < 0\}) \longrightarrow \mathbb{C}$$

> **Conjecture (Fact?):** This is a perfect pairing.

Maybe this is an obvious fact. I'm not very good at algebraic topology, but even working with some of the other students at Berkeley who are good at algebraic topology, we were unable even to directly calculate the dimension of $\mathrm{H}_n(\mathbb{C}^n, \{\Re(s) < 0\}) = \mathrm{H}_{n-1}(\{\Re(s) < 0\})$. On the other hand, I do know that:

> **Fact:** If s defines a smooth hypersurface in $\mathbb{CP}^{n-1}$, then $\dim(\frac{\mathbb{C}[x_1, \dots, x_n]}{(\frac{\partial}{\partial x_i} + \frac{\partial s}{\partial x_i} \cdot)}) = (d-1)^n$.

I'm much more of an algebraist than a topologist. The RH factor $\mathrm{H}_n(\mathbb{C}^n, \{\Re(s) < 0\})$ is the realm of topology and analysis, and is inaccessible to pure algebra, whereas the LH factor $\frac{\mathbb{C}[x_1, \dots, x_n]}{(\frac{\partial}{\partial x_i} + \frac{\partial s}{\partial x_i} \cdot)}$ is in the realm of pure algebra. So I will tell you how to completely analyze it.

Turning the question into homological algebra

We would like to understand the vector space $\mathbb{C}[x_1, \dots, x_n]/(\frac{\partial}{\partial x_i} + \frac{\partial s}{\partial x_i} \cdot)$. By "understand", I mean for example: $\mathbb{C}[x_1, \dots, x_n]$ has a distinguished basis (the monomials); we'd like a distinguished basis for the quotient, and an explicit description of the quotient map with respect to these bases.

Here's a useful problem-solving technique: "resolve" all quotients by finding them as H_0 of some chain complex.

In this case, I choose the following complex. Start with the graded commutative algebra $V_\bullet = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$, where $|\xi_i| = 1$ for homological degree.

$$V_\bullet = \mathbb{C}[x_1, \dots, x_n] \otimes \Lambda^\bullet(\xi_1, \dots, \xi_n)$$

I make it into a chain complex, *but not a dg algebra*, by choosing the differential

$$\partial_{\text{full}} = \underbrace{\sum_i \frac{\partial s}{\partial x_i} \frac{\partial}{\partial \xi_i}}_{\partial_{\text{cl}}} + \underbrace{\sum_i \frac{\partial^2}{\partial x_i \partial \xi_i}}_{\partial_{\text{Leb}}}$$

The graded commutative algebra $V_\bullet$ has a supergeometric interpretation as the algebra of (polynomial) functions on the *odd cotangent bundle* $\Pi T^*\mathbb{C}^n$. The ξ_is correspond to the coordinate functions vector fields $\frac{\partial}{\partial x_i}$, thought of as (linear) coordinate functions on the cotangent bundle. Why the cotangent and not tangent bundle? Algebraic reason: if you want to work equivariantly for the $\mathrm{GL}(n)$ action on $\mathbb{C}^n$, then you should let the ξ_is transform in the dual representation to how the x_is transform. Geometric reason: The (odd) tangent bundle to any manifold has a canonical (odd) vector field on it. The (odd) cotangent bundle to any manifold has a canonical (odd) (symplectic) Poisson bivector field on it. Here the Poisson bivector field is the principal symbol of the second-order operator ∂_{full}.

We want to understand the homology of $(V_\bullet, \partial_{\text{full}})$. The basic idea is to study this in pieces. In particular, we study $(V_\bullet, \partial_{\text{cl}})$, and then understand ∂_{Leb} as a perturbation of this. Our basic tool is an easy version of homotopy-transfer of algebraic structures (here the algebraic structure is "a choice of Maurer–Cartan element"):

Homotopy perturbation lemma (1960s): Suppose you are given a *retraction* (in any additive category!)

$$(H_\bullet, \partial_H) \underset{\phi}{\overset{\iota}{\leftrightarrows}} (V_\bullet, \partial) \circlearrowright \eta \qquad \begin{aligned} \iota\phi &= \mathrm{id}_H \\ \phi\iota &= \mathrm{id}_V - [\partial, \eta] \end{aligned}$$

and a *perturbation* $\partial \rightsquigarrow \partial + \delta$, so that $(\partial + \delta)$ is a new differential on $V_\bullet$. *Provided that* $(\mathrm{id}_V - \delta\eta)$ *is invertible*, you get a new retraction:

$$\big(H_\bullet, \tilde\partial = \partial_H + \iota \circ (\mathrm{id} - \delta\eta)^{-1}\delta \circ \phi\big) \underset{\tilde\phi = (\mathrm{id}-\eta\delta)^{-1}\circ\phi}{\overset{\tilde\iota = \iota\circ(\mathrm{id}-\delta\eta)^{-1}}{\leftrightarrows}} (V_\bullet, \partial + \delta) \circlearrowright \tilde\eta = \eta(\mathrm{id}-\delta\eta)^{-1}$$

Proof: Check a bunch of equations.

Corollary: Suppose that $H_\bullet$ is supported in degree 0. Then $\tilde\partial = 0$, and so the homology of V does not deform. Suppose furthermore that $V_\bullet$ is supported in nonnegative degrees. Then $\delta \circ \phi = 0$ (as its domain is in degree 0, so its codomain would be in degree -1), and so $\tilde\phi = \phi$.

Note that $(\mathrm{id} - \eta\delta)^{-1} = \mathrm{id} + \eta(\mathrm{id} - \delta\eta)^{-1}\delta$.

The most standard way to force $(\mathrm{id} - \delta\eta)$ to be invertible is to extend scalars by tensoring with $\mathbb{C}[\![\hbar]\!]$, for some formal variable $\hbar$, and then asking that $\delta = O(\hbar)$. Then $\delta\eta$ is pro-nilpotent, and so $(\mathrm{id} - \delta\eta)^{-1}$ exists. The next most standard thing is what we will do: there may be an $\mathbb{N}$-filtration on $V_\bullet$ for which $\delta\eta$ lowers degree. Then $\delta\eta$ is ind-nilpotent, and again $(\mathrm{id} - \delta\eta)$ is invertible. Filtrations are one of the standard ingredients in constructions of spectral sequences. The HPL is essentially a formula-full replacement of spectral sequences; it has the advantage of being completely deterministic.

As an application, consider setting $\partial = \partial_{\text{cl}}$ and $\delta = \partial_{\text{Leb}}$. We can set $H_\bullet = \mathrm{H}_\bullet(V_\bullet, \partial_{\text{cl}})$, with zero differential. We call the differential $\sum_i \frac{\partial s}{\partial x_i}\frac{\partial}{\partial \xi_i}$ by the name "cl" for "classical": in physics terminology, this is the differential that imposes "the classical equations of motion" (so you might say that ∂_{full} imposes "the quantum equations of motion"). Indeed, the complex $(V_\bullet, \partial_{\text{cl}})$ has a geometric interpretation: it is the *derived critical locus of s*. The point is that

$$\mathrm{H}_0(V_\bullet, \partial_{\text{cl}}) = \mathbb{C}[x_1, \dots, x_n]/(\tfrac{\partial s}{\partial x_1}, \dots, \tfrac{\partial s}{\partial x_n})$$

is the algebra of functions on the *critical locus* $\{\mathrm{d}s = 0\} \hookrightarrow \mathbb{C}^n$.

> **Fact:** By homogeneity, the critical/singular locus of s resides entirely at 0 if it is zero-dimensional, and otherwise it includes points at infinity. Thus if s defines a smooth hypersurface in $\mathbb{CP}^{n-1}$ then $\{\mathrm{d}s = 0\}$ is the origin with some multiplicity. Since each defining relation $\frac{\partial s}{\partial x_i} = 0$ is of degree $d-1$, and there are n of them, it follows from Bezout's theorem that this multiplicity is the origin in the critical locus is $(d-1)^n$. Put another way, $\dim \mathrm{H}_0(V_\bullet, \partial_{\text{cl}}) = (d-1)^n$.

In fact, all other homology groups of $(V_\bullet, \partial_{\text{cl}})$ vanish if s is smooth. The geometric explanation of this is that $\mathrm{H}_\bullet(V, \partial_{\text{cl}})$ is the algebra of functions on the *odd cotangent bundle* to the critical locus of s, but this critical locus is zero-dimensional. A little later I will prove that $\mathrm{H}_{>0}(V, \partial_{\text{cl}}) = 0$ when s is generic, but for now let's take it as true. Recall that smoothness is a generic condition — the discriminant of s, which is some complicated polynomial in the coefficients of s, must not vanish. In general, a *generic condition* is one that holds away from finitely many closed subvarieties, i.e. we will ask that finitely many polynomials in the coefficients of s do not vanish.

So we have $V_\bullet$ and $\partial = \partial_{\text{cl}}$, and we have $H_\bullet = \mathrm{H}_\bullet(V_\bullet, \partial_{\text{cl}})$, which generically is supported in degree 0. We have the projection map $\iota : V_0 = \mathbb{C}[x_1, \dots, x_n] \to \mathcal{O}(\{\mathrm{d}s = 0\}) = H_0$. (Why "$\iota$", by the way? Because it will deform to the "integration" map.) Let's choose a splitting ϕ for ι. The projection preserves polynomial degree, so let's ask that the splitting does as well. Indeed, $V_\bullet$ is naturally $\mathbb{N}$-graded by assigning the x_is weight 1 and the ξ_is weight $d-1$, so that ∂_{cl} has weight 0. Then we can always choose a homotopy η preserving this grading.

Now let's see what happens when we turn on $\delta = \partial_{\text{Leb}} = \sum_i \frac{\partial^2}{\partial x_i \partial \xi_i}$. It does not preserve the extra $\mathbb{Z}$-grading, but rather lowers it by d. So $\partial_{\text{Leb}}\eta$ is ind-nilpotent: for any given input f, eventually $(\partial_{\text{Leb}}\eta)^N f = 0$ for N/d larger than the maximal weight of f. Thus we have that:

> **Corollary:** If s is smooth, then $\mathrm{H}_\bullet(V, \partial_{\text{full}})$ is $(d-1)^n$-dimensional in degree 0, and vanishes otherwise.

Note moreover that $\tilde{\iota}$ is completely determined by the choice of $\tilde{\phi}$ and $\partial + \delta$, since $\tilde{\iota}$ must vanish on $\ker(\partial + \delta)$ and the image of $\tilde{\phi}$ is a complement to the kernel, and $\tilde{\iota}$ is then the unique map with the prescribed behavior on this complement. Thus we see in fact that:

> Any choice of splitting $\phi : \mathcal{O}(\{\mathrm{d}s = 0\}) \to \mathbb{C}[x_1, \dots, x_n]$ (a vector space map, not a map of algebras) determines an identification $\mathrm{H}_0(V, \partial_{\text{full}}) \cong \mathcal{O}(\{\mathrm{d}s = 0\})$. The choice of homotopy η is useful when looking for an explicit description of the projection $V_0 \to \mathrm{H}_0(V, \partial_{\text{full}})$.

Thus one fully understands (in the sense of explicit matrices for an explicit basis) the "Stokes" map $\mathbb{C}[x_1,\dots,x_n] \to \frac{\mathbb{C}[x_1,\dots,x_n]}{(\frac{\partial s}{\partial x_i}\cdot + \frac{\partial}{\partial x_i})}$ as soon as one fully understands the critical locus — one should choose a splitting as vector spaces of the restriction map $\mathbb{C}[x_1,\dots,x_n] \to \mathcal{O}(\{\mathrm{d}s = 0\})$, and although the choice doesn't matter one must also choose some homotopy η, at least in lowest degree.

Aside: We are primarily interested in the case when s is homogeneous. But in the real world it is inhomogeneous. So I'd like to make a few comments on this situation. Let's suppose first that s is a polynomial such that the highest-degree part s_{top} is smooth. Writing $s = s_{\text{top}} + s_{\text{sub}}$, we can incorporate the sub-leading terms as part of the deformation:

$$\partial_{\text{full}} = \underbrace{\sum_i \frac{\partial s_{\text{top}}}{\partial x_i}\frac{\partial}{\partial \xi_i}}_{\partial} + \underbrace{\sum_i \frac{\partial s_{\text{sub}}}{\partial x_i}\frac{\partial}{\partial \xi_i} + \sum_i \frac{\partial^2}{\partial x_i \partial \xi_i}}_{\delta}$$

Then we still have that this δ lowers the "polynomial grading" described earlier, and so we can still run the homotopy perturbation lemma.

Actually, this is also a technique for understanding the critical locus of an inhomogeneous s. We can turn on just s_{sub} and not ∂_{Leb} with HPL if we want.

The more standard way that people approach inhomogeneous actions s is to suppose that s is a non-singular quadratic plus a *higher order* deformation. The advantage of this approach is that when s is pure nondegenerate quadratic, then the "classical" homology is one-dimensional, and hence so is the homology of any deformation. But this makes it clear that if this higher-order deformation is turned on with finite amplitude, the whole story breaks: the homology must grow in dimension, but it cannot with homotopical deformations. So there are two ways to go, which are largely equivalent. One is to work over a formal power series ring $\mathbb{C}[\![\lambda]\!]$, where λ is some "coupling constant", and ask that $s = \text{nondegenerate quadratic} + O(\lambda)$. The other approach is to replace the polynomial ring $\mathbb{C}[x_i]$ with a formal power series ring $\mathbb{C}[\![x_i]\!]$, so that the higher-order corrections to ∂_{cl} are small. But then ∂_{Leb} is a very large correction. So you decide not to study integrals against $e^s\text{Leb}$ but rather agains $e^{s/\hbar}\text{Leb}$, where $\hbar$ is yet another formal variable. Then the pertinent differential is $\partial_{\text{full}} = \partial_{\text{cl}} + \hbar\partial_{\text{Leb}}$, and we have the necessary convergence.

In any case, the "integration" map $\tilde{\iota}$ has a natural expression in terms of Feynman diagrams. We explain this in more low-brow language in the paper with Owen. But briefly, the point is that the classical ι for pure-quadratic $s = \sum_{ij} s_{ij}x_ix_j/2$ evaluates every polynomial at 0, and the homotopy can be taken to be $\eta = \frac{1}{\deg}(s^{-1})_{ij}\xi_i\frac{\partial}{\partial x_i}$, so that $\delta\eta$ takes $f(x)$ either to $\sum (s^{-1})_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}$, which is like "closing a loop" or to $\sum (s^{-1})_{ij}\frac{\partial f}{\partial x_i}\frac{\partial t}{\partial x_j}$, where $t(x)$ are the "interaction terms", which is like "adding a vertex", and each of these operations is weighted by some number corresponding to a "symmetry factor". Iterating the procedure infinitely, and then keeping only the closed diagrams (constant terms of polynomials) gives the usual Feynman sum.

Completely explicit formulas from an ad hoc choice of basis

I've said that you can do this all explicitly, so I owe you an example. Suppose that in $s(x)$ all of the coefficients $a_i = s_{i\dots i}$ on the pure monomials $(x_i)^d$ are non-zero. Then we can write $s = s_{\text{diag}} + s_{\text{mix}}$, where $s_{\text{diag}}(x) = \sum_i a_i (x_i)^d/d!$. We want to compute the homology for $\partial_{\text{cl}} = \sum_i \frac{\partial s}{\partial x_i}\frac{\partial}{\partial \xi_i}$, and moreover compute it with a distinguished splitting $H_0 \to V_0$ and a distinguished homotopy. Let's try to to write $\partial_{\text{cl}} = \partial_{\text{diag}} + \partial_{\text{mix}}$ and run the homotopy perturbation lemma.

So let's look at the complex $(V_\bullet, \partial_{\text{diag}})$. The important observation is that as a complex (in fact, as a dgca) it factors as a tensor product:

$$\left(\mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_n],\ \sum_i a_i \frac{x_i^{d-1}}{(d-1)!}\frac{\partial}{\partial \xi_i}\right) = \bigotimes_i \left(\mathbb{C}[x_i,\xi_i],\ a_i \frac{x_i^{d-1}}{(d-1)!}\frac{\partial}{\partial \xi_i}\right)$$

Then consider the tensorand $\left(\mathbb{C}[x,\xi], a\,\frac{x^{d-1}}{(d-1)!}\frac{\partial}{\partial \xi}\right)$. Note that $\mathbb{C}[x,\xi] = \mathbb{C}[x] \oplus \mathbb{C}[x]\xi$, since ξ is odd. It is clear that the homology here is $(d-1)$-dimensional in degree 0. Indeed, the homology has a basis consisting of the images of x^m for $m < d-1$. Let's use this basis to split the homology $\phi : H_0 \to V_0$. Moreover we can choose a homotopy:

$$\eta(x^m\xi) = 0, \quad \eta(x^m) = \begin{cases} 0, & m < d-1 \\ \frac{(d-1)!}{a}\, x^{m-d+1}\,\xi, & m \geq d-1 \end{cases}$$

Then we can tensor together these choices. The homology $\mathrm{H}_0(V_\bullet, \partial_{\text{diag}})$ has as a basis the image of all monomials $x_1^{m_1}\cdots x_n^{m_n}$ with all $m_i < d-1$, and so is $(d-1)^n$-dimensional as we said earlier in the general case. This basis picks out a splitting $\phi : H_0 \to V_0$. There is in general no canonical way to tensor together chain-complexes-with-choice-of-homotopy. One good choice is to set $\eta_{\text{diag}}(x_1^{m_1}\cdots x_n^{m_n})$ to 0 if all $m_i < d-1$ and otherwise

$$\eta_{\text{diag}}\left(x_1^{m_1}\cdots x_n^{m_n}\right) = \frac{\sum_i \frac{\xi_i}{a_i}\left(\frac{\partial}{\partial x_i}\right)^{d-1}}{\sum_i \binom{m_i}{d-1}}\left(x_1^{m_1}\cdots x_n^{m_n}\right).$$

This then gives the first term $\eta : V_0 \to V_1$. In formulas this is the only part of η that will actually matter. So we can make some ad hoc choice how to extend it to the rest of $V_\bullet$. Since $V_\bullet$ split into a sum of complexes based on the total degree (degree in x) + $(d-1)$(degree in ξ), we can ask to extend η preserving this splitting.

Now we have all the ingredients we need to turn on $\delta = \partial_{\text{mix}}$ with the homotopy perturbation lemma. In general, we then need $(\mathrm{id} - \partial_{\text{mix}}\eta_{\text{diag}})$ to be invertible. By splitting $V_\bullet$ via the total degree, we see that $(\mathrm{id} - \partial_{\text{mix}}\eta_{\text{diag}})$ is block diagonal with finite-dimensional blocks. So invertibility consists of countably many polynomials in the coefficients of s (the determinants of the blocks) all not vanishing — such a condition is not "generic", but it is *very general*. (Note that when $s_{\text{mix}} = 0$ then it is invertible, so we're looking at a *nonempty* intersection of countably many Zariski-open sets, so since we are working over an uncountable field the intersecting is uncountable and dense.)

Thus, for each choice of coordinates $\{x_i\}$, we conclude that:

> Working very generally in the choice of s, the homology of $(V_\bullet, \partial_{\text{cl}})$ is supported in degree 0 and has as its basis the images of the monomials $x_1^{m_1} \cdots x_n^{m_n}$ with all $m_i < d-1$. Using these monomials to pick the splitting $H_0 \to V_0$, we can choose a homotopy η_{cl} whose $V_0 \to V_1$ part is
> $$\eta_{\text{cl}} = \eta_{\text{diag}}\big(\text{id} - \partial_{\text{mix}}\eta_{\text{diag}}\big)^{-1}$$
> where
> $$\partial_{\text{mix}}\eta_{\text{diag}}(x_1^{m_1}\cdots x_n^{m_n}) = \begin{cases} 0, \text{ all } m_i < d-1, \\ \dfrac{1}{\sum_i \binom{m_i}{d-1}} \displaystyle\sum_{\substack{i_1,\dots,i_{d-1},j \\ \text{not all equal}}} \frac{s_{i_1\dots i_{d-1}j}}{s_{j\cdots j}} \frac{x_{i_1}\cdots x_{i_{d-1}}}{(d-1)!} \left(\frac{\partial}{\partial x_j}\right)^{d-1} (x_1^{m_1}\cdots x_n^{m_n}). \end{cases}$$

Again note that $\partial_{\text{mix}}\eta_{\text{diag}}$ preserves the total degree of a polynomial in the xs, so invertibility can be handled degree-by-degree.

In fact, a little bit more can be said. We know the homology of $(V_\bullet, \partial_{\text{cl}})$ is $(d-1)^n$-dimensional if s is smooth. Let's suppose that the $\text{H}_0(V_\bullet, \partial_{\text{cl}})$ does not have the images of $\{x_1^{m_1}\cdots x_n^{m_n}, \text{ all } m_i < d-1\}$ as a basis. Then in homology there is a linear combination between these, and that is detected in the part of $V_\bullet$ with (degree in x) + $(d-1)$(degree in ξ) $\leq n$. So in fact only finitely many polynomials in the coefficients of s must be non-zero:

> For generic s, $\text{H}_0(V_\bullet, \partial_{\text{cl}})$, and hence $\text{H}_0(V_\bullet, \partial_{\text{full}})$, has as a basis the images of the monomials $\{x_1^{m_1}\cdots x_n^{m_n}, \text{ all } m_i < d-1\}$.

What can happen is that the proposed degree-0 part of η_{cl} can make sense for some degrees of the input, and then for some very large degree perhaps $(\text{id} - \partial_{\text{mix}}\eta_{\text{diag}})$ fails to be invertible. If we only care to make calculations for some bounded degree of input, then we don't care that this η_{cl} fails to work higher up: some homotopy η_{cl} must exist (since we only need it to exist at the level of chain complexes of vector spaces).

Finally, remember that what we're really after is the complex $(V_\bullet, \partial_{\text{full}})$ and in particular the projection $p_{\text{full}} : V_0 \to \text{H}_0(V_\bullet, \partial_{\text{full}})$. So, generically we can choose the bases to be: monomial basis for V_0, and images of $\{x_1^{m_1}\cdots x_n^{m_n}, \text{ all } m_i < d-1\}$ for H_0. Then we want formulas for p_{full} with respect to this basis. As we said earlier, turning on ∂_{Leb} with HPL always wins, because the filtration gives the guaranteed convergence. Working very generally if we want to input arbitrarily large-degree polynomials, and generically if we bound the degree of the input, we find out that:

> **Theorem:** Identify $H_0 = \text{H}_0(V_\bullet, \partial_{\text{full}})$ with the span of the monomials $\{x_1^{m_1}\cdots x_n^{m_n}, m_i < d-1\}$. Let $p_{\text{diag}} : V_0 \to H_0$ denote the map that is the identity on these monomials, and kills all polynomials divisible by some x_i^{d-1}. Then the projection $p_{\text{full}} : V_0 \to H_0$ is

given by:

$$p_{\text{full}} = p_{\text{diag}}\big(\text{id} - \partial_{\text{mix}}\eta_{\text{diag}}\big)^{-1} \sum_{\ell \geq 0} \Big(\partial_{\text{Leb}}\eta_{\text{diag}}\big(\text{id} - \partial_{\text{mix}}\eta_{\text{diag}}\big)^{-1}\Big)^{\ell}$$

where $\partial_{\text{mix}}\eta_{\text{diag}}$ is as above (and preserves polynomial degree), and

$$\partial_{\text{Leb}}\eta_{\text{diag}}\big(x_1^{m_1}\cdots x_n^{m_n}\big) = \begin{cases} 0, & \text{all } m_i < d-1 \\ \dfrac{1}{\sum_i \binom{m_i}{d-1}} \displaystyle\sum_i \frac{1}{a_i}\left(\frac{\partial}{\partial x_i}\right)^d \big(x_1^{m_1}\cdots x_n^{m_n}\big), & \text{else} \end{cases}$$

and in particular drops polynomial degree by d.

One can try to interpret these formulas diagrammatically. I don't really expect that it would be enlightening to do so.

In summary, what have we done? Given the data of a (generic) degree-d polynomial s, we have written explicit formulas, which are at worst rational functions in the coefficients of s, that express any integral of the form

$$\langle f \rangle = \int_\gamma f\, e^s \,\text{dLeb}$$

in terms of the $(d-1)^n$ values $\langle x_1^{m_1}\cdots x_n^{m_n}\rangle$, $m_i < d-1$. These formulas hold for any contour γ (provided $\Re(s) \to -\infty$ at the ends of the contour). If we only see the contours as living in a $\mathbb{C}$-vector space, then this is essentially optimal (if the conjecture about a perfect pairing is true).

A bit more work should be done, of course. In real life, the space of contours has an integral structure. And in real life there is a preferred contour: the real axis $\gamma = \mathbb{R}^n \hookrightarrow \mathbb{C}^n$. For usual physics problems, the coefficients of s are entirely pure-imaginary, and then (generically) the real axis does not give exponential decay, but it has a unique small deformation that does (and the integrals over $\mathbb{R}$ converge conditionally to the values computed on the deformed contour).

But these integrals are very hard. In the Gaussian case, it's easy: up to some constant that depends only on n, if s is homogeneous quadratic and negative definite, then $\langle 1 \rangle = \int_{\mathbb{R}^n} \exp(s)\,\text{dLeb} = 1/\sqrt{\det(-s)}$. Shamil worked out the corresponding formula for pure quartic in two variables:

$$\int_{\mathbb{R}^2} \exp(s)\,\text{dLeb} = I_2(s)^{-1/6}\ {}_2F_1\left(\begin{matrix} \frac{1}{12} & \frac{5}{12} \\ & \frac{1}{2} \end{matrix} \middle| \, 6\frac{I_2(s)^3}{I_3(s)^2}\right)$$

where ${}_2F_1$ is one of Gauss's hypergeometric functions, and $I_2(s)$ and $I_3(s)$ are the unique quadratic and cubic SL(2)-invariants of s, normalized so that the discriminant is $6I_2(s)^3 - I_3(s)^2$.

So this is an indication that the final step is hard.

Comments on the inverse problem, and other further research

So where are we? If you know s, and know the integrals of some small monomials, you know everything else. There are $\binom{n+d-1}{d-1} \sim n^{d-1}$ degrees of freedom in choosing a degree-d polynomial in n variables, and the contour is another $(d-1)^n$ degrees of freedom. For fixed n and d, the formulas we've given above do allow you to reconstruct this data from the values of $n^{d-1} + (d-1)^n$ integrals, at least up to solving some system of algebraic equations. But as n and d grow, this is not a practical answer. In particular, we'd really like to fix d but let $n \to \infty$. For $d > 2$ we have exponentially many contours and and algorithm that requires knowing the expectation values of all $\langle x_1^{m_1} \cdots x_n^{m_n} \rangle$, $m_i < d-1$ will not be useful.

I don't really expect that in the totally general case there would be anything better. There are some special cases, though, worth commenting on.

We can say things completely when $n = 1$, although it's a bit trivial there. More interestingly, when $n = 2$ and $d = 3$, then H_0 has no cubic basis vectors. Since $\partial_{\text{Leb}}\eta_{\text{cl}}$ preserves degree mod 3, we see get formulas expressing all $\langle \text{cubic} \rangle$s entirely in terms of $\langle 1 \rangle$, independently of the contour. This allows to recover s from the $\langle \text{cubic} \rangle$s. But this doesn't continue for larger n and d.

Another special case arises when we force some of the coefficients in s_{mix} to vanish. Specifically, a degree-d monomial is either a basis vector for homology, or it is of the form $x^{d-1}y$ or x^d for x and y some variables. In the former case, $\partial_{\text{mix}}\eta_{\text{diag}}$ kills it. We can force $\partial_{\text{mix}}\eta_{\text{diag}}$ to turn the latter case into the former case by forcing every term in s_{mix} divisible by some $(x_i)^{d-3}$ to vanish. Then on degree-d polynomials $\eta_{\text{diag}}\partial_{\text{mix}}\eta_{\text{diag}} = 0$, and so $p_{\text{full}} = p_{\text{diag}}(\text{id} + \partial_{\text{mix}}\eta_{\text{diag}})(\text{id} + \partial_{\text{Leb}}\eta_{\text{diag}}) = p_{\text{diag}}(\text{id} + \partial_{\text{mix}}\eta_{\text{diag}} + \partial_{\text{Leb}}\eta_{\text{diag}})$. This isn't that ad hoc of a restriction: lattice models in such that each vertex has at least four neighbors, and an interaction term that involves a product over all nearest neighbors. So it's possible that we'll find Wick-like results here.

As a final comment, even just understanding the "undergraduate calculus" problem that we described here is valuable. Most importantly, of course, is that we might be able to use these techniques to compute non-perturbative expectation values of Wilson loops in Chern–Simons theory. No promises yet, of course, but it might work.