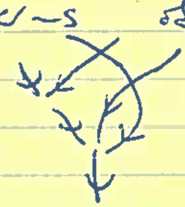
Ideals in derived algebra and boundary conditions in AKSZ-type field theories

Berkeley, Jan 27. Theo Johnson-Freyd

Thank you very much for the invitation. Everything in this talk is dg and over a field of characteristic zero.

I. Probably most of you know what is an operad, but I will review the basic idea. Operads parameterize algebras. The way they do this is as follows. Part of the data of an operad P is a collection of cochain complexes $P(n)$, $n \in \mathbb{N}$, of "n-to-1 operations" or "operations of arity n". Each complex should have an action of symmetric group S_n: if $(a,b) \mapsto a \star b$ is an operation in an algebra, $(a,b) \mapsto b \star a$ should also be. Moreover there are various composition maps $P(n) \otimes P(m) \to P(n+m-1)$, because surely $c \star (a \star b)$ is also an operation if $\star$ is. These can enforce algebra axioms like associativity or whatever. The general way to parameterize compositions is in terms of rooted trees



describes some composition $P(2) \otimes P(3) \to P(4)$.

The trees also tell you the "associativity" rules that composition and the S_n action should enjoy.

For this talk I will need to generalize the notion of operad in two simultaneous directions.

(1) A dioperad is like an operad except there are many-to-many operations and compositions are parameterized by (non-rooted) directed trees.

$P(m,n)$ = m-to-n operations. Various compositions $P(m_1,n_1) \otimes P(m_2,n_2) \to P(m_1+m_2-1, n_1+n_2-1)$.

Example: (a) There is a dioperad
LieB:
generated by operations Y, λ.
with relations $Y = -$ (twisted Y), $\lambda = -$ (twisted λ),

$$(\ldots) + \text{cyclic} = 0, \quad (\ldots) + \text{cyclic} = 0,$$

and $(\ldots) = (\ldots) + (\ldots) + (\ldots) + (\ldots)$.

(b) There is a dioperad Frob^{oco} generated by Y, λ, w/ (twisted Y) $= Y$, $\lambda =$ (twisted λ), $(\ldots) = (\ldots)$, $(\ldots) = (\ldots)$, $(\ldots) = (\ldots)$.

"oco" stands for "open and co-open". More generally, set $\mathrm{Frob}^{oco}_{d,d'}$ to be similar where $\deg(\lambda) = d$ and $\deg(Y) = d'$. Then

$H^\bullet$(any oriented manifold) is $\mathrm{Frob}^{oco}_{0,\dim(M)}$,

even if the manifold is open.

(2) An arrowed (di) operad is a 2-colored (di) operad with a distinguished operation

$$\begin{array}{c} \downarrow A \\ \bullet f \\ \downarrow B \end{array}.$$

Here's the reason for the name. An algebra for a (di) operad is a representation of it: a cochain complex V and, for every operation a multilinear map, such that composition in P = composition of multilinear maps.

An algebra for an arrowed dioperad is automatically an arrow $A \xrightarrow{f} B$.

. Examples:

Suppose P is an operad. Set

$$P^{\to}(m,n;m',n') = \begin{cases} P(m;1), & (m,n;m',n') = m,0;1,0 \\ P(m+n;1), & m,n;0,1 \\ 0 & \text{else}. \end{cases}$$

Then strict $P^{\to}$ algebras <u>are</u> homomorphisms of P-algebras.

<u>Main example:</u>

If P is any dioperad, set

$$P^{\triangleleft,\text{strict}}(m,n;m',n') = \begin{cases} 0 & \text{if } m=n'=0 \\ P(m+n;m'+n') & \text{else}. \end{cases}$$

i.e. you ignore the ~~colors~~ color except you <u>disallow</u> all-B-in-to-all-A-out.

III. Quasifree etc.

It is technically convenient although not strictly necessary, for disallow operations with $m=0$ or $n=0$, and to ~~say that~~ only allow identities + the unit only when $m+n=m'+n'=1$.
I call such dioperads <u>oco</u>.

An oco arrowed dioperad is <u>quasifree</u> if it is free as a $\mathbb{Z}$-graded (not nec. dg) object on a generating complex $G = \{G(m,n;m',n'),\ (m+n)(m'+n')\geq 2\}$

Standard arguments provide a model category structure on oco objects in which weak equivalences are q-isos and fibrations are surjections.
In this model, quasifree $\Rightarrow$ cofibrant and every ... has a quasifree resolution.

Suppose P is quasifree non-arrowed on generators G.
Then there is a quasifree arrowed dioperad $P^{\triangle}$
~~P~~ generated by

$$G^{\triangle}(m,n;m',n') = \begin{cases} 0 & m=n'=0 \\ G(m+n;m'+n')\,[m+n'-1] & \text{else.} \end{cases}$$

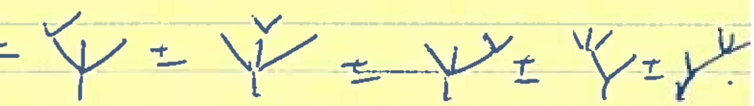
The differential is given by a sum over internal colorings.

<u>Theorem</u>: $P^{\triangle} \xrightarrow{\sim} P^{\triangle,\mathrm{str}}$.

Let me try to explain $P^{\triangle}$ by way of example.
Let's set $P = A_\infty$. Recall this is generated by operations μ_n, $n \geq 2$, with no symmetry rules.

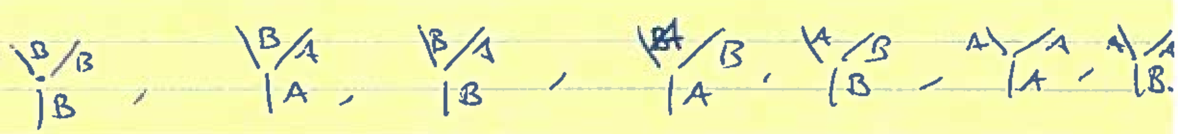
μ_3 is the associator, ...

So, writing $\widetilde{Y} = \mu_n$, we have

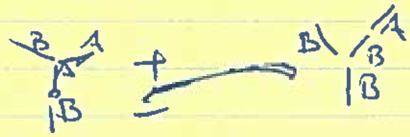
$\partial \mu_3 =$  , $\partial \mu_4 =$ 

etc.

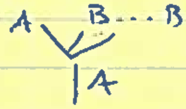
Then in $A_\infty^{\triangle}$, we have various operations:

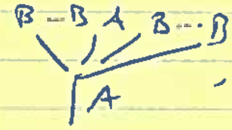


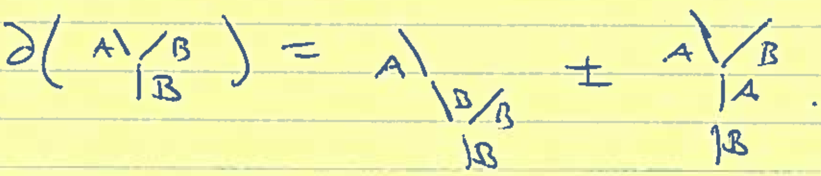
$\partial\left({}^{B}{}_{B}{}^{B}\right)=0$, $\partial\left({}^{B}{}_{A}{}^{A}\right)=0$, $\partial\left({}^{A}{}_{A}{}^{B}\right)=0$, but

$\partial\left({}^{B}{}_{B}{}^{A}\right) =$ 

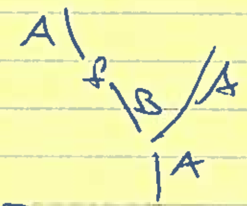
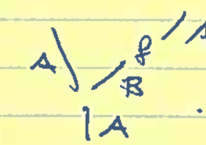
Look at the all-Bs part. It makes B into an A_∞-algebra.

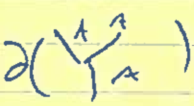
If you look at , it makes A into an A_∞ B-module

actually, if you look at , it makes A into an (A_∞-) B-bimodule.



In general, the  make $f: A \to B$ into a bimodule map.

But now we have  and .

$\partial($$)$ = this difference.

If f were an injection, then these would agree. So we make the difference exact

All together, $A_\infty^\triangle$-algebras are: an A_∞-algebra and some type of ideal.

<u>Here's a sharper statement</u>:

<u>Thm</u>: Suppose P is an operad. Then there is an equivalence

(~~homotopy~~ homotopy) $P^\triangle$-algebras $\simeq$ homotopy $P^\to$ algebras.

covering

$$\begin{array}{ccc} \text{homotopy } P^{\triangleleft}\text{-algebras} & \simeq & \text{homotopy } P^{\to}\text{ algebras} \\ \big\downarrow \text{forget} & & \big\downarrow \text{forget} \\ \{\text{arrows}\} & \xrightarrow[\text{of exact triangles.}]{\text{rotation}} & \{\text{arrows}\} \end{array}$$

$$\left(A \xrightarrow{f} B\right) \longmapsto \left(B \to \mathrm{cone}(f)\right).$$

So this really justifies thinking of $P^{\triangleleft}$-algebras as <u>P-ideals</u>.

Actually, even part of the theorem should surprize you: I'm telling you that if $A \xrightarrow{f} B$ is a $P^{\triangleleft}$-algebra, then in particular $\mathrm{cone}(f)$ is a P-algebra, at least when P is an operad. This <u>almost</u> holds in the more general dioperad case:

<u>Thm:</u> If $A \xrightarrow{f} B$ is a $P^{\triangleleft}$-algebra for P a dioperad, then $\mathrm{cone}(f)$ is a

$$P \boxtimes \mathrm{Frob}_{0,1} \text{ algebra.}$$

$P \boxtimes \mathrm{Frob}_{0,1}$ is almost P, just with some different degrees. Here $\boxtimes$ is the <u>Bordman-Voght tensor product</u> defined by

$$\cancel{P(m,n)}$$
$$(P \boxtimes Q)(m,n) = P(m,n) \otimes Q(m,n).$$

Its defining property is that if A is a P-alg and X is a Q-alg, then $A \otimes X$ is a $P \boxtimes Q$-alg.

<u>Example:</u> $\mathrm{LieBi} \boxtimes \mathrm{Frob}_{d,d'}$ is the version of LieBi where the bracket has degree d and the cobracket d' (up to a sign error).

IV. Oh, this reminds me: have you ever wondered how rotation of exact triangles interacts with tensor product? See, if

$$\cancel{A} \quad A \to B \qquad \text{and} \qquad X \to Y$$

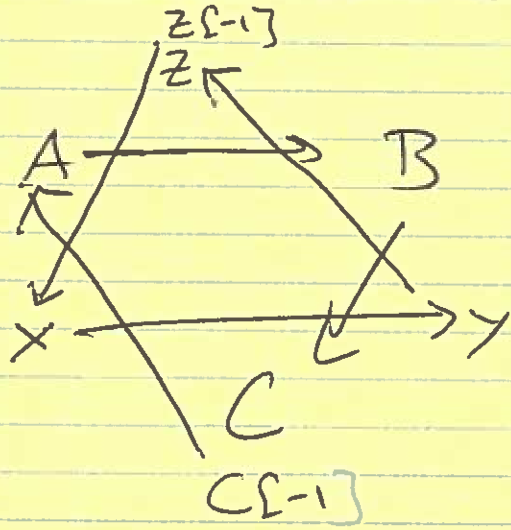
are arrows, then ~~A~~

$$A \otimes X \to B \otimes Y$$

is an arrow, but certainly the tensor product of exact triangles is not exact. ~~How~~
~~so ⊗ is not so ..., what~~
But if you have two exact triangles

$$A \to B \to C \quad \text{and} \quad X \to Y \to Z,$$

you can draw them as an <u>exact Star of David</u>.



and ~~then~~

<u>Exercise</u>: $\mathrm{Cone}(\rightrightarrows_{\otimes}) \simeq \mathrm{Cone}(\swarrow\otimes\swarrow) \simeq \mathrm{Cone}(\nwarrow\otimes\nwarrow)$.

This is axiom T3 in May's defn of "sym ⊗ ~~.....~~ triangulated category".
~~so ....~~

<u>N.B.</u>: One triangle is ↻, the other ↺, so "forward" rotation of one ~~..~~ must be accompanied by "backward" of the other.

Correspondingly you can tensor arrowed diagrams ~~dopends~~ do get dropped.

so that if $A \to B$ is P and $X \to Y$ is Q, then $\mathrm{cone}(A \otimes X \to B \otimes Y)$ is $P \boxtimes Q$.

$\underline{\mathrm{V}}$. Any world = algebras, operads, ...
has an operation called the Bar Dual.
It is defined, up to degree conventions, by

$$\mathbb{D}P = \text{q-free thing on } \underline{P}^* \text{ [shift somehow]}$$
with ∂ encoding composition in P.

Where $\underline{P} = P$ w/o identities.
For me it's convenient to shift ~~[illegible]~~ via

$$P^*(m,n;\, m',n')\,[m+n'-2].$$

It's called duality because $\mathbb{D}^2 P \xrightarrow{\sim} P$.

You might know the following universal property for ordinary operads: if P is an operad, $\mathbb{D}P$ is universal for a map

~~$\mathbb{D}P$ [illegible] is a m.~~

$$L_\infty \longrightarrow \cancel{\mathbb{D}P \otimes} P \otimes \mathbb{D}P$$

~~I.e. if A is a P alg and Q a $\mathbb{D}P$ alg then $A \otimes Q$ is L_∞, and $\mathbb{D}P$ is universal~~

$\underline{\text{Thm:}}$ For arrowed dioperads, $\mathbb{D}P$ is universal s.t. equipped with a map

$$LB_\infty \to P \boxtimes \mathbb{D}P$$
$$\| $$
$$\mathbb{D}\,\mathrm{Frob}^{oco}_{0,1}$$
$$\| $$
∞-version of Lie bialgebras.

Example: By construction,

$$(\mathbb{D}P)^{\triangleleft} \simeq \mathbb{D}(P^{\triangleleft}).$$

VI. I will skip quadratic duality. This story is just like in other situations. If you know what it means for an operad (or algebra or whatever) to be Koszul, then you know the arrowed drop. version.

Thm: If P is Koszul, so is $P^{\triangleleft, \mathrm{str}}$

~~This makes it easier easy to~~
Koszulity is a tool for computing minimal resolutions.

VII. Relative Poincaré duality

Now I can get to maybe the central example.

Suppose M is a compact, oriented d-dim manifold. Then of course $H^\bullet(M)$ is a $\mathrm{Frob}_{o,d}$-algebra: you have the cup product, but also a coproduct coming from Poincaré duality.

What if M has boundary? Then you have a map of commutative algebras

$$\cancel{H^\bullet(M) \to H^\bullet(\partial M)}.$$

$$H^\bullet(M) \to H^\bullet(\partial M)$$

Also ~~you can look at relative cohomology, and you get a map of~~ coalgebras

$$H^\bullet(M, \partial M) \to H^\bullet(M).$$

Look at relative cohomology $H^\bullet(M;\partial M)$.

Poincaré duality says $H^\bullet(M;\partial M) \simeq H_\bullet(M)$ [shift]

~~and so~~ $H^\bullet(\partial M) \simeq H_\bullet(\partial M)$ [shift]

and in fact

$$H^\bullet(\partial M) \longrightarrow H^\bullet(M;\partial M)$$

is a map of cocom coalgs (up to shift).

<u>More is true</u>: In fact,

$$H^\bullet(M;\partial M) \longrightarrow H^\bullet(M)$$

is a $\mathrm{Frob}^{\Delta,\,\mathrm{str}}_{0,2+}$ algebra.

This lifts to (de Rham) cochain level. model

$H^\bullet(M) \simeq \Omega^\bullet_{dR}(M)$,

$H^\bullet(M;\partial M) \simeq \Omega^\bullet_{\substack{\text{cpt}\\ \text{support}}}(M)$.

<u>Defn:</u> A <u>quasi-local</u> many-to-many operator on de Rham cochains is a family of homotopy [illegible] operators parameterized by ~~an~~ <u>energy scale</u> ~~length~~ <u>energy scale</u> E such that as $E \to \infty$ the support (of the integral kernel) becomes close to the diagonal.

Δ-inequality $\Rightarrow$ these form an (curved) dioperad

? whether the forms may end on ∂M.

Thm (Cochain-level relative Poincaré duality):

There is a unique map

$$\{\text{homotopy } \mathrm{Frob}^{\partial}_{0,2}\} \to \left\{\begin{matrix}\Omega\text{-loc operators} \\ \text{on } M\end{matrix}\right\}$$

s.t. the composition

$$\text{homotopy } \mathrm{Frob}^{\partial}_{0,2} \to \{\Omega\text{-loc operators}\} \xrightarrow{\varepsilon = 1} \mathrm{End}(\Omega_{cpt}(M) \to \Omega(M))$$

$$\xrightarrow{\sim} \mathrm{End}(H(M;\partial M) \to H(M))$$

induces the Frob^{∂}-str on H^{∂}.

VIII - Here's another example. Let me tell you about LB_∞-algebras. V. I will ignore degree shifts.

Then V is an L_∞-alg and a L_∞-coalg in a compatible way, ~~or~~

Take just V as an L_∞ alg. Then

$$\widehat{\mathrm{Sym}}(V^*[1]) = CE^\bullet(V, [\,,\,])$$

↑ power series

is an cdga. It is the "functions" on an "infinitesimal dg manifold" w/ coordinate chart V.

V^* is a L_∞-alg. The compatibility says:

$CE^\bullet(V, [\,,\,])$ is an L_∞-alg,

where the L_∞-operations on V^* are extended as multiderivations, i.e.

$CE^\bullet(V, [\,,\,])$ is a $\mathrm{hty}\,\mathrm{Po}_{3}$ algebra, i.e. "the infinitesimal manifold V"

is <u>Poisson</u>.

There are some degree shifts.

Suppose Z is Poisson inf manifold aka ~~...~~ (homotopy) LB algebra.

What is a <u>coisotropic</u>? Should have

$$Y \longrightarrow Z \quad \text{a "sub" (i.e. map of Lie algs)}$$

and

$$\ker\big(\mathcal{O}(Z) \to \mathcal{O}(Y) \big) \hookrightarrow \mathcal{O}(Z)$$
$$CE^{\cdot}(Z) \to CE^{\cdot}(Y)$$

a sub Lie alg. Unpacking:

$$Z \longrightarrow X \doteq \mathrm{cone}(Y \to Z)$$ should be a map of L_∞-<u>co</u>algebras.

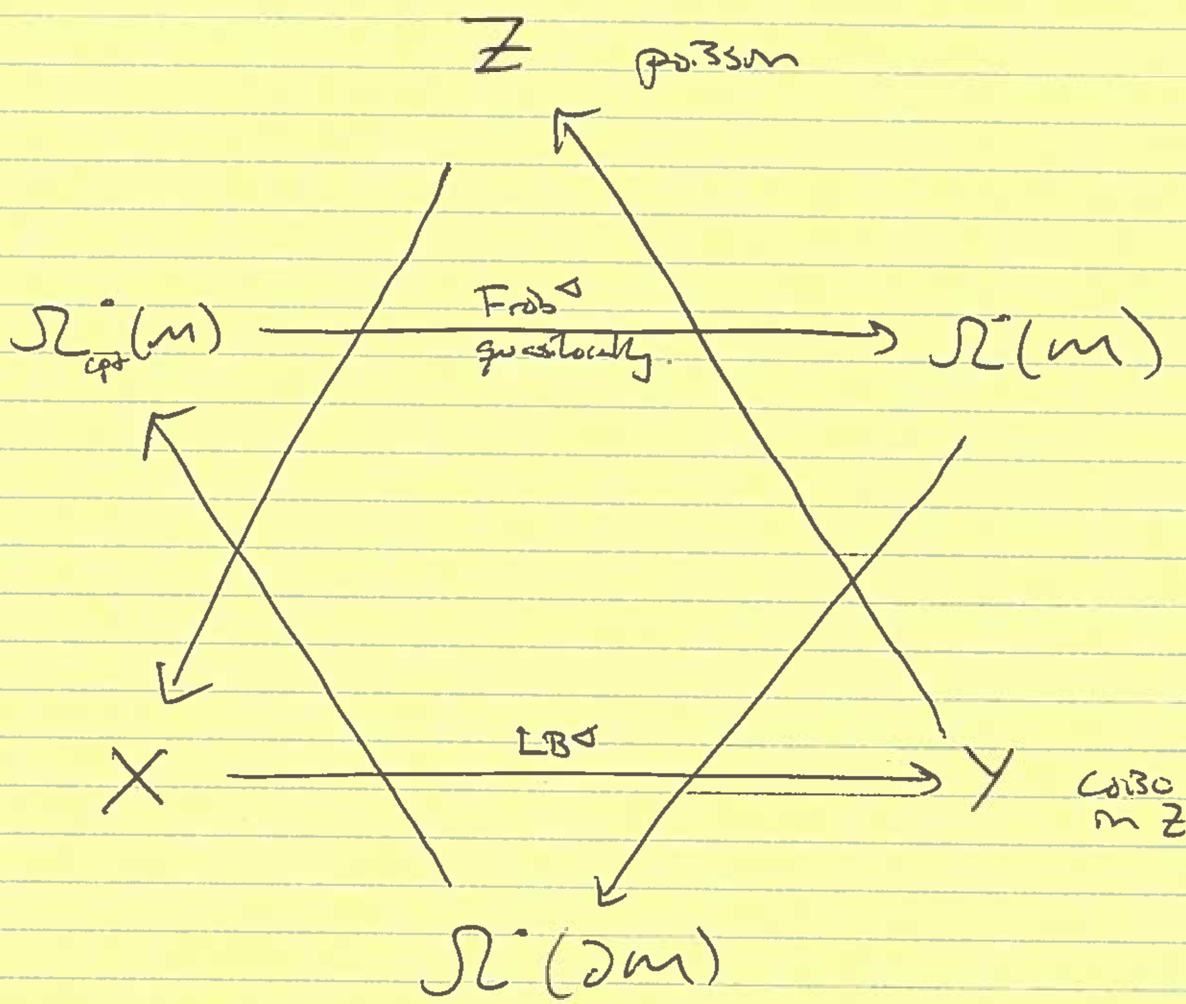
$$\begin{tikzcd} & Z \arrow[dl, "\substack{\text{Lie}\\\text{coalg}\\\text{map}}"'] & \\ X \arrow[rr] & & Y \arrow[ul, "\substack{\text{Lie alg}\\\text{map}}"'] \end{tikzcd}$$

<u>Proposed Defn:</u> ~~Under the ... Let Z~~ A map $Y \xrightarrow{f} Z$ is <u>coiso in Poisson</u> iff

$\mathrm{cone}(f) \to Y$ is an LB^{∂}-(homotopy-) algebra.

~~i.e. the data of~~

IX. Poisson AKSZ with coiso boundary.



<u>Exercise</u>: The invariant V-space — the tensor product of the exact ✡-of-David — is also

$$\Omega^\bullet(M)\otimes Z \times^h_{\Omega^\bullet(\partial M)\otimes Z} \Omega^\bullet(\partial M)\otimes Y$$

$$\simeq$$

$$= \mathrm{maps}(M, Z) \times^h_{\mathrm{maps}(\partial M, Z)} \mathrm{maps}(\partial M, Y)$$

$$= \mathrm{maps}\Big(\quad$$  $$\Big).$$

Since $LB^{\partial} \simeq \mathbb{D}\,Frob^{\partial}$ (since $LB \simeq \mathbb{D}\,Frob$),

this invariant space is LB, i.e. (shifted) Poisson.

This gives a (shifted) Poisson structure on the space of fields in the bulk-boundary system (of the expected shift).

The Poisson structure satisfies appropriate locality - condition. (Becomes local in UV limit.)

<u>N.B:</u> This is classical, not <u>quantum</u> field theory.
Dioperads $\leftrightarrow$ <u>tree-level</u> Feynman diagrams.