Advanced integration by Parts

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0. Someone once said that quantum field theory is that part of mathematics concerned with the evaluation of "oscillatory" integrals, i.e. those of the form

$$\langle f \rangle =: \int_{\mathcal{X}} f(x) \exp\left(\tfrac{i}{\hbar} s(x)\right) dx$$

where $f \in \mathcal{O}(\mathcal{X})$ is an "observable", $s \in \mathcal{O}(\mathcal{X})$ is the "action", $\hbar$ is "Planck's constant", and $\mathcal{X}$ is the "space of fields". What makes this difficult is that in QFT, $\mathcal{X}$ is typically

- infinite-dimensional
- super
- stacky.

and an analytic definition of $\int dx$ does not exist. Thus the goal is an algebraization of integration.

I will focus my talk on the case where $\int$ can be defined: $\mathcal{X} = \mathbb{A}^n$. I will absorb i into $\hbar$, thinking of $\hbar$ as "pure imaginary". I will call the coordinates on $\mathcal{X}$ $x_1, \dots, x_n$. The volume form should be the translation-invariant one.

Main idea 1: The main idea is that, by integration-by-parts,

$$\int \frac{\partial}{\partial x_i}\left(g \exp\left(\frac{s}{\hbar}\right)\right) = \cancel{0}\ \text{boundary term.}$$

Let's suppose we have some way of knowing the boundary term vanishes. Then for each i, g we have a

<u>Ward identity</u>:

$$\langle \hbar \partial_i g + g\, \partial_i s \rangle = 0.$$

We can package the ward identities into a complex:

$$T = \mathcal{O}^{\oplus n} \longrightarrow \mathcal{O}$$
$$\vec{g} = (g_1, \dots, g_n) \longmapsto \sum_i (\hbar \partial_i g_i + g_i \partial_i s) = \vec{g}[s] + \hbar \operatorname{div}(g)$$

<u>Main idea 2:</u> In many situations, H_0 of this complex is manageable. (In the best situations it is one-dimensional(!)) And $\langle\,\rangle : \mathcal{O} \to$ ground field factors through H_0.

The <u>algebraic part</u> of integration is to compute the map $\mathcal{O} \to H_0$. The <u>transcendental part</u> is the map $H_0 \to$ numbers.

1. The first situation where we know there is no boundary term is when $X = \mathbb{A}^n_{\mathbb{R}}$, s is real and smooth, $\hbar$ is pure imaginary, s grows not-too-slowly, and f, g grow not-too-quickly. Then the boundary term vanishes.
[Moreover, <u>stationary phase approximation</u> says:

If f has compact support and s does not have a critical point in $\operatorname{sup}(f)$, then $\langle f \rangle = O(\hbar^{\infty})$. ~~But when~~

This suggests to work over $\mathbb{C}[[\hbar]]$.

modulo $\hbar$, the ward identity is just $\langle g\, \partial_i s \rangle = 0$, ~~which is~~ and $\mathcal{O}/\langle g\, \partial_i s \rangle$ is the Jacobi ring. So:

If f vanishes on $\operatorname{crit}(s)$, then $\langle f \rangle = O(\hbar)$

(assuming $\langle f \rangle$ has an asymptotic expansion).

(stationary

③

This is what's really called the stationary phase approximation.

As a simplifying assumption, let's suppose s has a nondegenerate critical point at ~~the~~ origin:

$$s(x) = \frac{1}{2} \sum a_{ij} x_i x_j + b(x)$$

$b = O(x^3)$. Then a is invertible.

Exercise: ~~Let $f(\cdot)$~~, ~~Then~~ $\exists\, g_1, \ldots, g_n$ s.t.

$$f = f(0) + g_i \frac{\partial s}{\partial x_i} \pm \cancel{f}$$

Suppose we can find such g. Then

$$\langle f \rangle = \langle f(0) \rangle + \sum \langle g_i \, \partial_i s \rangle$$

$$= \langle f(0) \rangle - \hbar \sum \langle \partial_i g_i \rangle \qquad (*)$$

+ contributions from outside a nbhd of the origin.

To make those contributions go away, let's decide

$$\mathcal{O} = \mathbb{C}[[x_1, \ldots, x_n]].$$

Now let's solve the exercise.

$$f - f(0) = f^{(1)}_k x_k + \frac{1}{2} f^{(2)}_{k\ell} x_k x_\ell + \ldots$$

$$\stackrel{!}{=} g_i(x) a_{ij} x_j + g_i \, \partial_i b + \ldots$$

Set $\eta = a^{-1}$. Note that $\partial_i b = O(x^2)$.

then $$f - f(0) = \underbrace{f^{(1)}_{k} \eta_{ki} a_{ij}}_{\delta_{kj}} x_j + f^{(2)}_{k\ell} x_k x_\ell + \cdots$$

so $$g_i(x) = f^{(1)}_k(x)\,\eta_{ki} + O(x).$$

To calculate the next term, we might as well assume $f = f^{(2)} x^2/2 + \cdots$ is quadratic. Then

$$\frac{1}{2} f^{(2)}_{k\ell} x_k x_\ell + \cdots = \left(\frac{f_{k\ell} x_k \eta_{\ell i}}{2}\right) a_{ij} x_j + \cdots$$

Rather than tracking the bs, let's rewrite (*).

~~By ... find ... $\langle f \rangle = \langle f(0)\rangle +$~~

Let's look for an operator $\eta : \mathcal{O} \to T$ so that

$$f = f(0) + \eta_i(f)\, a_{ij} x_j .$$

One solution:

$$\eta(\text{constant}) = 0$$

$$\eta_i(f) = \frac{1}{N} \eta_{ij} \partial_j f, \qquad f \text{ homogeneous of degree } N.$$

There are other solutions, but this is the only one preserving the grading by poly degree. Then (*) gives:

$$\langle f \rangle = \langle f(0) \rangle + \langle \eta_i(f)\, a_{ij} x \rangle$$

$$= \langle f(0) \rangle + \langle \eta_i(f)\, \partial_i s \rangle - \langle \eta_i(f)\, \partial_i b \rangle$$

$$= \langle f(0) \rangle - \underbrace{\hbar \langle \partial_i [\eta_i(f)] \rangle}_{\text{higher deg. in } \hbar} - \underbrace{\langle \eta_i(f) \partial_i b \rangle}_{\text{higher deg in } x}.$$

Does this help? Yes:

Cor: ~~Suppose $\langle f \rangle$ has an asymptotic expansion~~

Thus iterating $(**)$ converges in power series to something in ~~$\mathcal{O}$~~ $\mathbb{C}[[\hbar]]$.

Cor: If $\langle f \rangle$ has an asymptotic expansion, it must be given by this limit.

Problem: Suppose we don't know much analysis, and so don't know that $\langle f \rangle$ has an expansion. ~~Does $(**)$ nevertheless~~ Is there nevertheless a unique power series? Maybe you'd use other ward identities — do you get the same answer?

To see what's going on, look back at the chain complex we have

$$\begin{tikzcd} T \arrow[r] & \mathcal{O} \arrow[l, bend right, "\eta"'] \end{tikzcd}$$

and two different differentials: $\vec{g} \overset{\partial_0}{\longmapsto} g_i a_{ij} x_j$

and $\partial = \partial_0 + \delta$,

$$\delta(\vec{g}) = g_i \partial_i b + \hbar \partial_i g_i .$$

Then

$$H_0(T \xrightarrow{\partial_0} \mathcal{O}) = \mathbb{C}[[\hbar]] = \mathbb{K}$$

This is witnessed by η. Set

$$p : \mathcal{O} \to \mathbb{K}, \quad f \mapsto f(0),$$

$$i : \mathbb{K} \longrightarrow \mathcal{O} \quad 1 \longmapsto 1.$$

Then $pi = id$ and $ip = id - [\partial, \eta]$ in degree 0.

δ is a <u>perturbation</u> of the differential.

Iterating (**) ~~says~~ says that after perturbation, we get

$$\tilde{p} : \mathcal{O} \to \mathbb{K}$$

commuting w/ ∂_1 viz

$$\tilde{p} = p + p\circ(-\delta\eta) + p(-\delta\eta)^2 + \dots$$
$$= p(\mathrm{id} + \delta\eta)^{-1}.$$

This converges since $\delta\eta$ raises ~~power~~ power series degree.

Well, almost. It says this <u>if</u> $H_0(T \xrightarrow{\partial} \mathcal{O}) \cong \mathbb{K}$.

What we need to know is that the elements of T that were ∂_0-closed don't start messing things up. ~~W~~ What are the ∂_0-closed elements? ~~they are of the form~~

Answer: write

$$T \wedge T \xrightarrow{\partial_0} T \xrightarrow{\partial_0} \mathcal{O}$$

$$\partial_0(g \wedge h) = (\partial_0 g)\cdot h - g\,\partial_0 h.$$

This complex is exact in ~~the~~ degree 1.

Does η extend? Yes. Consider

$$\Lambda^\bullet T = \mathbb{C}[\![x_1, \dots, x_n, \underbrace{\xi_1, \dots, \xi_n}_{\text{degree } 1}, \hbar]\!].$$

Then $\partial_0 = a_{ij}\, x_i \frac{\partial}{\partial \xi_i}$.

$$\delta = \frac{\partial b}{\partial x_i}\frac{\partial}{\partial \xi_i} + \hbar \frac{\partial^2}{\partial x_i \partial \xi_i}$$

and $\eta = \begin{cases} 0, & N=0 \\ \frac{1}{N}\, \eta_{ij}\, \xi_i \frac{\partial}{\partial x_j}, & N \neq 0 \end{cases}$

where N is the total degree.

<u>Homological perturbation Lemma:</u>

A <u>retraction</u> is

$$\eta \circlearrowright V \underset{i}{\overset{p}{\rightleftarrows}} H$$

where V, H are complexes, p, i are chain maps,

$$pi = id, \quad \text{and} \quad ip = id - [\partial_V, \eta]$$

A <u>perturbation</u> is $\partial_V = \partial_0 \rightsquigarrow \tilde{\partial}_V = \partial_0 + \delta$.

It is <u>small wrt η</u> if $(id + \delta\eta)^{-1}$ exists.

Given a retraction, define

$$\tilde{p} = p(id + \delta\eta)^{-1}$$
$$\tilde{i} = i(id + \eta\delta)^{-1} i = i + \eta(id + \delta\eta)^{-1}\delta i$$
$$\tilde{\eta} = \eta(id + \delta\eta)^{-1}$$
$$\tilde{\delta}_H = p(id + \delta\eta)\delta i.$$

Then $\tilde{\eta} \circlearrowright (V, \partial_0 + \delta) \underset{\tilde{i}}{\overset{\tilde{p}}{\rightleftarrows}} (H, \partial_H + \tilde{\delta}_H)$

is a retraction.

<u>Proof:</u> Pure algebra.

<u>Cor:</u> $H_0(T \xrightarrow{\partial_0 + \delta} \mathcal{O}) \simeq \mathbb{K}$.

2. So much for perturbative integrals. Let's look at nonperturbative ones. Then

$$\mathcal{O}(\mathcal{X}) = \mathbb{C}[x_1, \ldots, x_n]$$

polynomials. Nonperturbatively, I can absorb $\hbar$ into s.

I still have my twisted de Rham complex

$$\Lambda^\bullet T, \quad \partial = \frac{\partial s}{\partial x_i}\frac{\partial}{\partial \xi_i} + \frac{\partial^2}{\partial x_i \partial \xi_i}.$$

$$\| \\ \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$$

corresponding to integrals of shape

$$\int f \exp(s)$$

for f, s both polynomials.

When can I assure the integral converges and there are no boundary terms? I need to choose a contour γ of integration; convergence is when

$$\mathrm{Re}(s) \ll 0 \text{ on the ends of } \gamma.^*$$

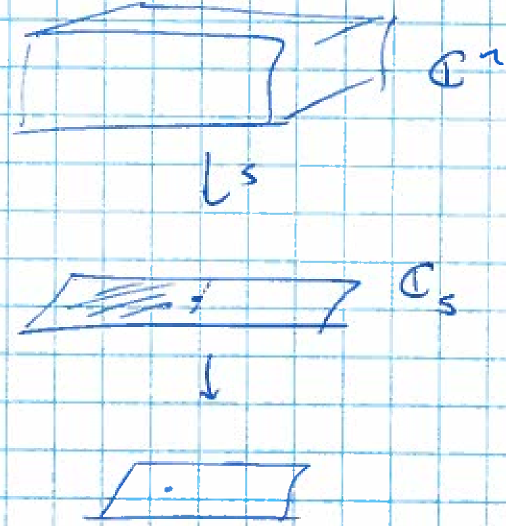
Then, provided γ stays in the convergent part, we can move it around w/o changing the integral. So

$$\{\text{contours}\} = H_n(\mathbb{C}^n, \{\mathrm{Re}(s) \ll 0\})$$

Let's understand this space. Suppose $s = s^{(d)}$ is homogeneous of degree d. Suppose also that the corresponding hypersurface $\{s=0\} \subseteq \mathbb{CP}^{n-1}$ is smooth.

* plus regularity-of-γ conditions that don't affect the homology.

Then we can contract $\{Re(s) << 0\}$ to $s = -1$.



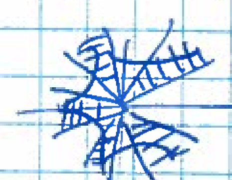
Since s is homogeneous, the fibers of $s = \cancel{\text{const}}\ a$ are all the same, $a \neq 0$. LES:

$$\begin{tikzcd} H_n(\mathbb{C}^n, \{Re(s) << 0\}) \arrow[r] & H_n(\mathbb{C}) = 0 \\ H_{n-1}(\{Re(s) << 0\}) \arrow[u] & \end{tikzcd}$$

Anyway, an Euler characteristic calculation verifies

$$\dim H_n(\mathbb{C}^n, \{Re(s) << 0\}) = (d-1)^n.$$

e.g. $n=1$, $d=5$



This still works if

$$s = s^{(d)} + \text{lower order}$$

with $\operatorname{disc}(s^{(d)}) \neq 0$.

What about the twisted de Rham complex?

The strategy of writing $\partial = \partial_0 + \delta$ was quite successful. Well,

$$\partial = \frac{\partial s}{\partial x_i}\frac{\partial}{\partial \xi_i} + \frac{\partial^2}{\partial x_i \partial \xi_i} = \frac{\partial s^{(d)}}{\partial x_i}\frac{\partial}{\partial \xi_i} + \frac{\partial(\text{l.o.})}{\partial x_i}\frac{\partial}{\partial \xi_i} + \frac{\partial^2}{\partial x_i \partial \xi_i}$$

In polynomial degree, $\frac{\partial s^{(d)}}{\partial x_i}\frac{\partial}{\partial \xi_i}$ dominates.

Choose a "polynomial" grading by setting

$$wt(x)=1, \quad wt(z) = d-1.$$

Then $wt\Big(\underbrace{\frac{\partial s^{(d)}}{\partial x_i}\frac{\partial}{\partial z_i}}_{\partial_0}\Big)=0,$

and $\delta = \frac{\partial(\ldots)}{\partial x_i}\frac{\partial}{\partial z_i} + \frac{\partial^2}{\partial x_i \partial z_i}$ ~~is small~~

is small in the polynomial filtration.

~~∂_0 is the~~

$V=(\Lambda^\bullet T, \partial_0)$ is the Koszul complex for $s^{(d)}$. Since $s^{(d)}$ is smooth,

$$\{d s^{(d)}=0\}$$

is a complete intersection. Therefore

$$V \xrightarrow{p} H_0 = \mathcal{O}(\{d s^{(d)}=0\})$$
$$= \frac{\mathcal{O}}{(\partial_i s^{(d)})}$$

is a quasi-isomorphism. We can choose i, η

$$\eta \circlearrowright V \underset{i}{\overset{p}{\rightleftarrows}} \text{Jac}$$

to preserve the polynomial degree. There is some ambiguity in this, but no matter.

Now run HPL. Get $\tilde{p}$, etc.

N.B.: $\tilde{\imath} = (\mathrm{id} + \eta\delta)^{-1} i = i - \eta\delta\, i + \cdots$

lowers degree

$= i$.

We have proved:

Thm: For each grading-preserving splitting i

$$\begin{tikzcd} \mathcal{O}(\mathbb{C}^n) \arrow[d, shift left, "p"] \\ \mathcal{O}(\{d s^{(d)} = 0\}) \arrow[u, shift left, "i"] \end{tikzcd},$$

there is a unique isomorphism

$$\mathcal{O}(\{d s^{(d)} = 0\}) \cong H_0(\partial + \delta)$$

" twisted de Rham homology for $s = s^{(d)} +$ lower ord.

such that i splits the canonical projection

$\mathcal{O}(\mathbb{C}^n) \to H_0(\partial + \delta)$.

In terms of $\mathcal{O}(\{d s^{(d)} = 0\})$, the projection is

$$\tilde{p} = p(\mathrm{id} + \delta\eta)^{-1}$$

for any choice of grading-preserving homotopy η

(i.e. $ip = \mathrm{id} - [\partial_0, \eta]$)

Finally, it is a nontrivial fact that the pairing

$$\{\text{contours}\} \times \{\text{twisted de Rham homology}\} \xrightarrow{\int} \mathbb{C}$$

is perfect. The Theorem says that to go from $\mathcal{O}(\mathbb{C}^n) \to \{\text{twisted homology}\}$ is pure algebra, and the rest is purely transcendental.

Example: If $s^{(d)} = a_1 x_1^d + a_2 x_2^d + \dots + a_n x_n^d$
$+$ ~~=~~ generic

with all $a_i \neq 0$, then you can perturb around the diagonal s, and find:

Cor: The classes of

$$\langle x_1^{m_1} \cdots x_n^{m_n} \rangle, \text{ all } m_i \leq d-2$$

are a basis of quantum observables.

Warning: Corollary fails for

$$x^4 + 2x^3y + 2xy^3 + y^4.$$

3. At its most basic, what we used about the twisted de Rham complex was:

- It was filtered, w/ associated graded commutative

In fact, I used a bit more for explicit computations:

- ignoring the differential, ~~$(V, d) = V$~~ it was a graded com. alg.

i.e. "associated graded" didn't do anything to the underlying v-space.

In the examples, we moreover had

- ∂ is a 2^{nd}-order diff. operator

but that is not essential.

You probably know that if A is a filtered assoc. algebra w/ $\mathrm{gr}\, A$ commutative, then $\mathrm{gr}\, A$ is Poisson. Similarly, if A is a filtered complex w/ $\mathrm{gr}\, A$ commutative, $\mathrm{gr}\, A$ is P_0.

$\uparrow$ Poisson w/ $\deg \{,\} = -1$.

We can think of this as:

> The P_0 object ~~Spec~~ $(\mathrm{gr}\, A)\}$ is the "classical limit" of the "quantum" object A.

<u>Conversely</u>:

> Given a P_0 object (w/ appropriate gradings), <u>BV quantization</u> asks for a quantum filtered thing w/ the P_0 thing as its classical limit.

Thus, to ~~explore~~ construct interesting quantum systems, one option is:

- Construct an interesting P_0 system.
- Try to quantize it.

Return to perturbative world. Let's decide that, ignoring the differentials,

$$A = \mathrm{gr}\, A = \widehat{\mathrm{Sym}}(\cancel{V} L)$$

"linear functions"

You probably know that:

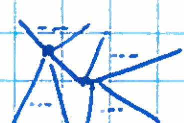
dg ca str on A $\longleftrightarrow$ L_∞-coalg. structure on $L[1]$

Defn. A (semistrict homotopy) $Pois_0$ structure on A is an L_∞ structure on $A[-1]$ such that every multibracket is a multiderivation.

Thus $\mu_m : (A[-1])^{\otimes m} \to A[-1][k-2]$ is determined by its values on L. What we have on L are:

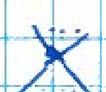
for every m, n, a map
$$Sym^m L \to Sym^n L$$
of degree -1. (~~m-th bracket, n-th~~ Taylor
$\mu_m(l_1, \dots, l_m) \in A$, nth Taylor coef.

The L_∞ relations become:

$$\sum_{\text{binary trees}}$$  $$= 0 \qquad (\star)$$

After ruling out ·, ·, · terms, ~~...~~ I can separate out · as ∂_L = linear differential.

Then $(\star)$ becomes

$$[\partial_L,$$  $$] = \sum$$  $$(\star\star)$$

where now · also · is ruled out.

Who knows about bar complexes? If you think ~~ab~~ how

$$CE^\circ : \text{Lie alg} \to \text{dg com alg}$$

goes, you recognize that the generators solve

$$\partial(\text{generator}) = \text{quadratic}.$$

Equation (~~☆~~☆) is of this type.

<u>Defn.</u>

Operads : dioperads :: rooted trees : directed trees

In this world, (☆☆) defines

$$\text{Cobar}(\text{co Frobenius}) =: \mathbb{D}\text{Frob}$$

commutative frobenius algebras. Plays the role of Com.

<u>Remark:</u> I will not use it, but I remark that

$\mathbb{D}\text{Frob}$ is a cofibrant resolution of

"$\text{Lie Bi}_{-1,-1}$" where

the subscripts indicate that $\deg(\text{~~bra~~}) = \deg(\text{cobracket}) = -1$

You can also define Pois_n, by adjusting some degrees. In terms of a linear clef, it is

$$\mathbb{D}\text{Frob}_n$$

comult has degree 0, mult has degree n.

E.g. $H_\bullet M$ is Frob_n if M is oriented + $\dim = n$.

AKSZ construction:

If M is compact + oriented and n-dim and X is ~~n-shifted~~ Pois_n and symplectic,

then $\mathrm{maps}^{\mathrm{loc}}(M, X)$
$$\shortparallel$$
$$\Omega^\bullet M \otimes X$$
is ~~$\mathrm{Pois}_{n-\dim M}$~~ Pois_0.

~~Idea~~: Construction uses integration on M: ~~Eg~~

~~Eg~~ pull back ω_X to $M \times \mathrm{maps}(M, X)$, integrate out M directions.

~~Variation~~

Generalization:
If A is (homotopy) Frob_n, L is DFrob_n, then $A \otimes L$ is DFrob_0.

$\boxed{\text{Everything in derived sense.}}$

Is $\Omega^\bullet M$ (homotopy) Frob_n? Of course:

$\Omega^\bullet M \simeq H^\bullet M$, use transfer.

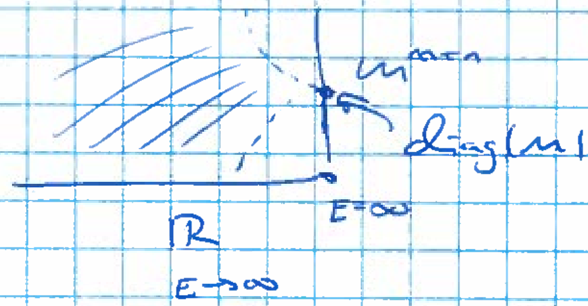
But this tends to produce highly nonlocal operations.

Defn: A local operation on $\Omega^\bullet M$ is ~~a~~ a family of operations, parametrized by energy scale E, s.t. as $E \to \infty$, integral kernel is supported near $\Delta(M)$.

An m-to-n gluing operation
i.e. it is an integral kernel on

$$M^m \times M^n \times \mathbb{R}$$

~~suppor~~ ~~which~~ with support like



These compose (Δ ineq).

Thm: $\Omega^\bullet M$ has a unique (up to contractible choices) hoFrob structure in which all operations are quasilocal.

At diagonal = free level.

What about quantum version?

Defn: A (semistrict homotopy) BD structure on

$$A = \widehat{\mathrm{Sym}}\, L\, [[\hbar]]$$

is a differential ∂ s.t.

∂ is an nth order diff. op. mod $\hbar^n$.

In terms of diagrams, expanding in $\hbar$ and thinking about how diff ops compose gives

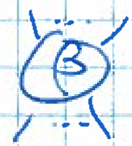
s.t. $\beta = \beta_1 + \beta_2$ + sum over graph

TCB is a (genus-graded) properad

~~properad~~ operad : properad ::

rooted trees : directed connected graphs w/ no directed cycles.

So Then "BD" is $\mathbb{D}\mathrm{Frob}_0$ in the properad sense,

because  $\longleftrightarrow$ frobenius operation w/ genus β.

e.g.  $\longleftrightarrow$  $\in$ Frob.

Still have:

If A is a Frob_n and L is $\mathbb{D}\mathrm{Frob}_n$,

then $A \otimes L$ is $\mathbb{D}\mathrm{Frob}_0$

now in the properad sense.

Turns out:

For odd n,  $= 0$ in Frob_n by sign rules,

so $\underset{\text{dioperad sense}}{\mathbb{D}\mathrm{Frob}_n} = \underset{\text{properad sense}}{\mathbb{D}\mathrm{Frob}_n}$.

Suppose $\Omega^{\bullet}_{cpt}R$ has a quasilocal hoFrob_1-structure in the properad sense.

Suppose L is all in degree 0, w/ a Poisson
$\parallel$
P_0, $\mathbb{B}_1$ structure

Then
$$\Omega^{0,*}_{cpt}(\mathbb{R}) \otimes L[\![t]\!]$$ is

quasilocally ~~[illegible]~~ BD.

It deforms $(\Omega^{*}_{cpt}\mathbb{R} \otimes L, \, \partial_{dR} + \partial_L)$.

For each ~~[illegible]~~ bump function, you can choose an explicit contraction

$$\eta \circlearrowright \Omega^{0,*}_{cpt}\mathbb{R} \otimes L \underset{i}{\overset{p}{\rightleftarrows}} L$$

i = insert at bump function.

And applying $\widehat{\mathrm{Sym}}$, and dividing by N, get

$$\eta \circlearrowright \mathrm{Sym}(\Omega_c \mathbb{R} \otimes L) \underset{i}{\rightleftarrows} \mathrm{Sym}(L)$$

Deforming this $\mathbb{B}$ as before: For each bump function, get

$$\tilde{\eta} \circlearrowright \mathrm{Sym}(\Omega_c \mathbb{R} \otimes L) \underset{\tilde{i}}{\overset{\tilde{p}}{\rightleftarrows}} \mathrm{Sym}(L).$$

all in degree zero, so no differential.

In fact, $\tilde{p}$ does not depend on choice of bump function, and $\tilde{i} = i$.

Since ∂ is quasilocal,
modulo high degree in power series,

$\Rightarrow$ $\partial($ anything supported in U $)$

$$\cap$$

is supported in some larger nbhd.

So choose U, V far away.

$$\begin{array}{c} \mathrm{Sym}\, L \\ \downarrow \tilde{\tau}_1 \\ U \end{array} \qquad\qquad\qquad \begin{array}{c} \mathrm{Sym}\, L \\ \downarrow \tilde{\tau}_2 \\ V \end{array}$$

$$\Downarrow$$

multiply in $\mathrm{Sym}(\Omega^\bullet_c \mathbb{R} \otimes L)$

$$\downarrow \tilde{p}$$

$\mathrm{Sym}\, L$.

Since U, V are far apart and ∂ is glue,

Thm: this composition is indep of $\tilde{\tau}_1, \tilde{\tau}_2$ and defines an associative multiplication on $\mathrm{Sym}\, L$ deforming the Poisson structure.

Note: ~~[illegible]~~ This allows $\dim L = \infty$.

~~So~~ Punchline: A graph-level cochain Poincaré duality on $\mathbb{R}$ gives a wheel-free Kontsevich formality.

I claimed a while back I had such a Poincaré duality. Unfortunately I had a subtle error. In fact:

<u>Thm</u>: There is a non-zero obstruction in genus 2 to graph level quasi-local Poincaré duality.

<u>Remark</u>: If the Poisson structure on Sym(L) is constant + linear, you only need $\wedge$ multiplication of $\Omega_\bullet(\mathbb{R})$.

You recover Moyal + Kog...