# $\star$-quantization via lattice topological field theory

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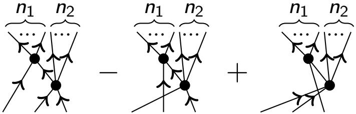
These slides available at
`math.berkeley.edu/~theojf/slides-2013-6-18.pdf`

Preprint available at
`math.berkeley.edu/~theojf/StarQuantization.pdf`

## Definition

A *Poisson formal manifold* is vector space $V$ (over $\mathbb{Q}$) and a Poisson structure $\pi$ on $\widehat{\mathrm{Sym}}(V)$. Equivalently, Taylor coefficients  : $V^{\wedge 2} \to \mathrm{Sym}^n(V)$ satisfying, for every $n$:

$$0 = \sum_{n_1+n_2=n} \frac{n_2+1}{\binom{n}{n_1}} \sum_{\substack{\text{shuffles of the}\\ \text{outgoing strands}}}$$



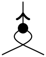

## Remark

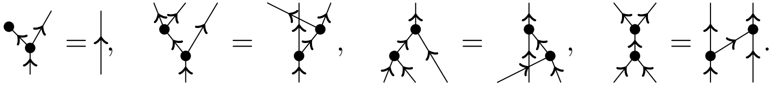
This is an example of a *properad* PoisF: compositions make sense along any connected directed acyclic graph, or *dag* for short.

## Goal

Construct associative multiplication $\star = \odot + \hbar\frac{\pi}{2} + \hbar^2 a_2 + \dots$ on $\widehat{\mathrm{Sym}}(V)[\![\hbar]\!]$; all $a_i$s bidifferential operators in $\mathbb{Q}[\pi^{ij}, \partial_k \pi^{ij}, \dots]$.

## Definition

The properad Frob$_1^o$ of *open(=nonunital) one-shifted commutative Frobenius algebras* has generators  and  $=$  of degree 0 and  $= -$  of degree $-1$, and relations

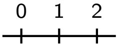


The bar/cobar construction gives a cofibrant replacement hFrob$_1^o$ freely generated by arbitrarily-complicated dags modulo symmetric group actions, with complicated differential.

## Example

Homology $\mathrm{H}_\bullet(S^1, \mathbb{Q})$ is a Frob$_1^o$-algebra.

## Theorem

*$\mathrm{hFrob}_1^o$ acts on $\mathrm{C}_\bullet = \mathrm{CellularChains}_\bullet(\cdots$*  *$\cdots)$, lifting the action on $\mathrm{H}_\bullet$, by translation-invariant operators that are* quasilocal*: for each generator there exists $\ell$ such that only cells within distance $\ell$ interact. The space of such actions is contractible.*

## Proof.

Since  $= 0$ by symmetry, $\mathrm{Frob}_1^o$ is Koszul, giving a minimal cofibrant replacement $\mathrm{shFrob}_1^o$ and a map $\mathrm{hFrob}_1^o \to \mathrm{shFrob}_1^o$. To build the $\mathrm{shFrob}_1^o$-action (and prove contractiblity), check by hand that a short finite list of obstructions vanish; the rest of the obstructions vanish by computing the homology of the complex of translation-invariant quasilocal operators. □

Given PoisF-algebra $V$, let $\Delta_{m,n,g} : (\mathrm{C}_\bullet \otimes V)^{\otimes m} \to (\mathrm{C}_\bullet \otimes V)^{\otimes n}$ be:

$$\Delta_{m,n,g} = \sum_{\substack{\text{dags } \Gamma \text{ with } m \text{ inputs,} \\ n \text{ outputs, and genus } g}} (\text{combinatorial term}) \times$$

$$\times \, (\Gamma \text{ as generator of } \mathrm{hFrob}_1^o) \otimes (\Gamma \text{ as composition in PoisF})$$

Extend to an $m$th-order differential operator on $\widehat{\mathrm{Sym}}(\mathrm{C}_\bullet \otimes V)$.

## Lemma

*The operator* $\Delta = \sum_{m,n,g} \hbar^{g+m-1} \Delta_{m,n,g}$ *on* $\widehat{\mathrm{Sym}}(\mathrm{C}_\bullet \otimes V)[[\hbar]]$ *satisfies* $(\partial + \Delta)^2 = 0$*, is* $O(\hbar)$*, and vanishes on* $\mathrm{Sym}^i(\mathrm{C}_\bullet) \otimes \mathrm{Sym}^i(V) \subseteq \mathrm{Sym}^i(\mathrm{C}_\bullet \otimes V)$.

## Corollary

*For* $z \in \mathbb{Z}$, insertion $\iota_z : \widehat{\mathrm{Sym}}(V)[[\hbar]] \to \big(\widehat{\mathrm{Sym}}(\mathrm{C}_\bullet \otimes V), \partial + \Delta\big)$ *at* $z$ *is a quasiiso. Its left inverse* $p$ *is unique and* $z$*-independent, and given explicitly by the Homological Perturbation Lemma.*

## Definition

The *star-product* $\star : \widehat{\mathrm{Sym}}(V)[[\hbar]] \otimes \widehat{\mathrm{Sym}}(V)[[\hbar]] \to \widehat{\mathrm{Sym}}(V)[[\hbar]]$ is defined modulo high powers of $V, \hbar$ by:

$$\star = p \circ \odot \circ (\iota_{z_1} \otimes \iota_{z_2})$$

$z_2 - z_1 > \ell$; $\ell > 0$ depends on the powers of $V, \hbar$ that you want.

## Main Theorem

*$\star$ is a well-defined associative deformation of $\odot$, and is independent of the choice of* $\mathrm{hFrob}_1^o$ *action used. It satisfies all requirements to be a universal star product.*

## Proof.

Well-definedness, associativity, and independence are formal, and use that $\Delta$ vanishes on symmetric-times-symmetric. To check other requirements involves combinatorics of diagrams. □

## Definition

The *classical TFT of AKSZ type* with target $X$ assigns to a spacetime $M$ the derived space of locally-constant maps $M \to X$, called $\underline{\mathrm{Maps}}(M_{\mathrm{dR}}, X)$. If $M$ is an oriented $d$-dimensional manifold and $X$ has a symplectic form of homological degree $-k$, then $\underline{\mathrm{Maps}}(M_{\mathrm{dR}}, X)$ has a symplectic form of homological degree $d - k$.

In infinite dimensions, symplectic forms do not invert. And yet:

## Generalization

*If $X$ has a $k$-shifted* homotopy *Poisson structure, then* $\underline{\mathrm{Maps}}(M_{\mathrm{dR}}, X)$ *has a $(k-d)$-shifted homotopy Poisson structure. Choosing this amounts to choosing a $d$-shifted open homotopy Frobenius algebra structure on* $\mathrm{Chains}_\bullet(M)$ *at the dioperadic level.*

## Definition

Dioperads are like properads, but only use tree-level compositions.

For every oscillating integral, there is a *BV complex*: 0-cycles are gauge-invariant observables and 0-boundaries are Ward identities. In the classical limit, you get the *derived critical locus*, a dg space with $(-1)$-shifted homotopy Poisson structure.

## Definition

A *BV quantization* of a $(-1)$-shifted homotopy Poisson structure is a deformation of the dg structure that matches the Poisson structure to first order. For derived critical loci, this is the same as constructing a measure.

## Lemma

*Homotopy* $(-1)$*-shifted Poisson* $=$ dioperadic $\mathrm{Bar}(\mathrm{Frob}_0^o)$.
*BV quantization* $=$ properadic $\mathrm{Bar}(\mathrm{Frob}_0^o)$.

($\mathrm{Frob}_0^o$ $=$ open 0-shifted commutative Frobenius (pr/di)operad.)

## Definition

A *factorization algebra* $F$ encodes the derived (i.e. BV–BRST) space of observables of a QFT: for every open neighborhood $U$ in spacetime, $F(U)$ is a chain complex, and for every $U_1, \dots, U_n \subseteq U$ pairwise disjoint (to enforce Heisenberg uncertainty), there is a multiplication map $\bigotimes F(U_i) \to F(U)$. (+locality axioms)

## Fact (Francis, Lurie)

*Framed n-dim topological factorization algebras = $E_n$ algebras.*

My construction is not local, so $\left(\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet(U) \otimes V)[\![\hbar]\!], \partial + \Delta\right)$ is not a factorization algebra. But quasilocality $\Rightarrow$ locality in the "large-volume limit" = take the lattice spacing very small.

Standard constructions from quantum field theory, and how they appear here:

| | |
|---|---|
| Derived space of classical fields | $\left(\mathrm{Cochains}^\bullet(\mathbb{R}) \otimes V^*, \partial\right)$ |
| Algebra of quantum observables | $\left(\widehat{\mathrm{Sym}}(\mathrm{Chains}_\bullet \otimes V)[\![\hbar]\!], \partial + \Delta\right)$ |
| Renormalization scheme | choice of higher homotopies in $\mathrm{hFrob}_1^o$-action |
| effective action integrating out modes inside each interval | $\Delta = \sum (\Gamma \in \mathrm{hFrob}_1^o) \otimes (\Gamma \in \mathrm{PoisF})$ |
| $n$-point function | $n$-fold $\star$-product |
| Path integral | Homological Perturbation Lemma |