# von Neumann algebras are O(2) fixed points ... somehow

Theo Johnson-Freyd and David Reutter

Subfactors and Applications,
MFO, 28 July 2025

This talk contains no new results. Our goal is to draw attention to an old question.

## Outline + Punchline:

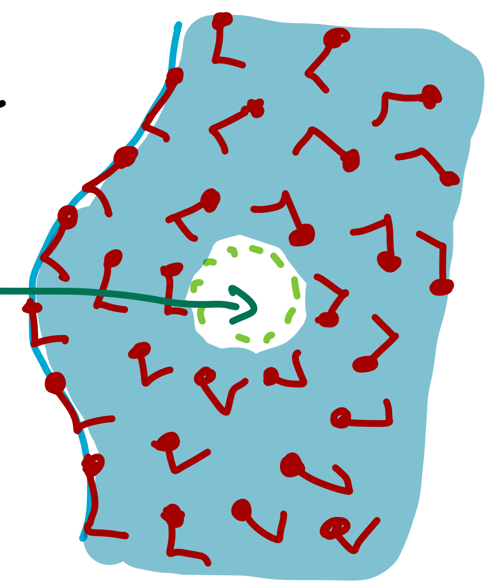
1. Framed 2D TFTs with boundary
   $\rightsquigarrow$ f.d. associative algebras
2. Unoriented 2D TFTs with boundary
   $\rightsquigarrow$ $*$-Frobenius algebras
3. Framed 2D QFTs with boundary
   $\rightsquigarrow$ associative algebras
4. $O(1)\times O(1)$ 2D QFTs with boundary
   $\rightsquigarrow$ von Neumann algebras (!)
5. $O(2)$ 2D QFTs with boundary
   $\rightsquigarrow$ von Neumann algebras (!!)

# 1. Framed 2D TFTs with boundary conditions

A (wick-rotated) quantum field theory can answer questions like "what is the expection value of ..."?

Some axioms for framed 2D TFT with b.c.:

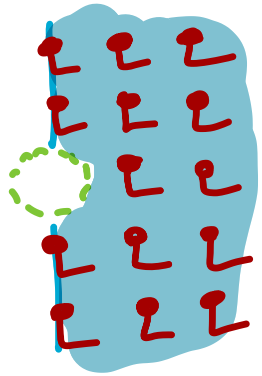
- "questions" are expressed by pictures.
- "pictures" are 2D manifolds with glazed boundaries. (Glue doesn't stick to glazing.)
- questions like "what are all valid insertions at this point?" take values in vector spaces.

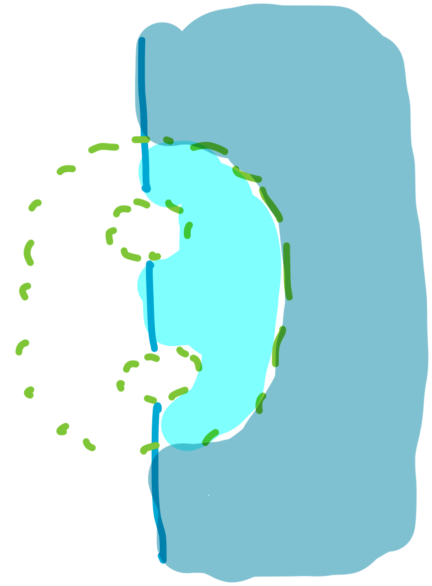


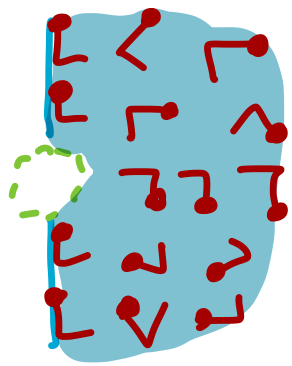
- all states are framed.

  Near glazed boundary, framing is product.

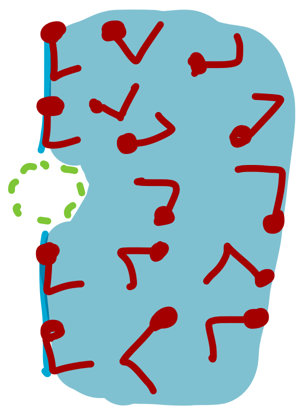
# Some interesting vector spaces of boundary operators

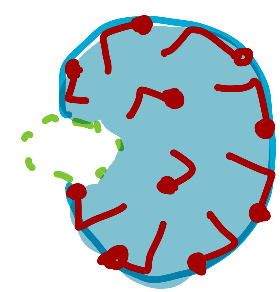
$A =$ 

Inclusions like: 

$S_1 =$ 

make $A$ into an associative algebra and $S_{\pm 1}$ into $A$-bimodules.

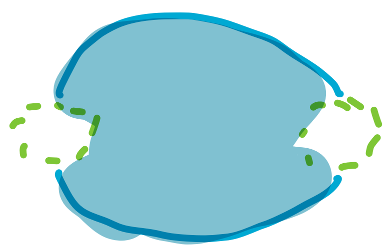
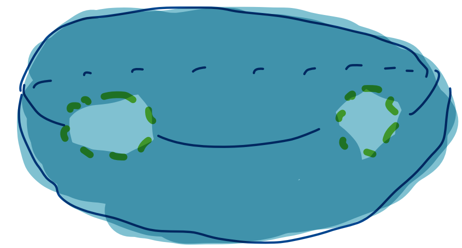
$S_{-1} =$ 

Here is a trace on $S_1$: 

# Remote Dectability Axiom:

"Anything that is consistent with all other operators is a valid operator."

In TFTs, this implies that pairings like

 and 

are duality pairings. E.g. $S_1 \cong A^\vee$.

It also implies gluing rules, e.g.

$S_1 \otimes_A S_{-1} \to A$ is an iso.

will explain later.

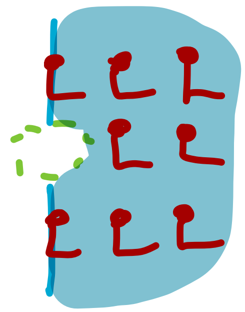
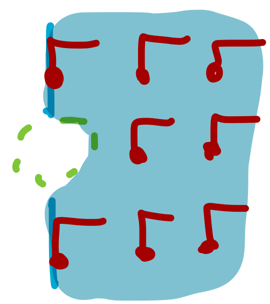
This is how to translate to the story of TFTs that you may already know.

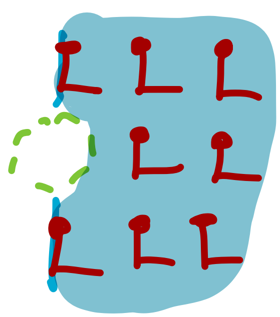
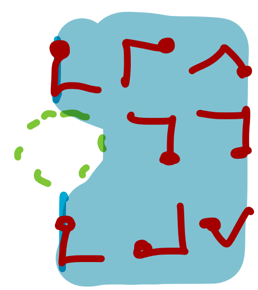
## 2. Unorienting the story

Consider the tangential structure of:

- no structure at the boundary
- no structure in the bulk
- near the boundary, the no structure is product.

Get new nontrivial isomorphisms:

$A \simeq$  $\simeq$  $\simeq A$

$A \simeq$  $\simeq$  $\simeq S, \simeq A^\vee$

These make $A$ into a $*$-Frobenius algebra.

## Why is this "the answer"?

Unorienting the bulk and boundary have to do with the groups $O(2)$ and $O(1)$, and in topology, we like to describe groups by their classifying spaces.

The data and properties of "$*$-Frobenius algebra" match exactly the low-dimensional cells in standard cell models of $BO(1)$ and $BO(2)$.

So topology says we are done with the characterization of unoriented 2D TFTs + b.c.

## 3. Framed QFTs on flat spacetime.

Actual QFT is not topological. It lives on flat manifolds. Sometimes but not always it has more:

- extension to curved manifolds is called "coupling to background gravity"
- integrating over the choice of curved manifold is called "dynamical gravity"

If you don't couple to gravity, then all you have are Poincaré symmetries.

Framed case: just translation.

At glazed boundaries, the rule remains: local structures, inc. metrics, are product.

# Flowing to the deep UV.

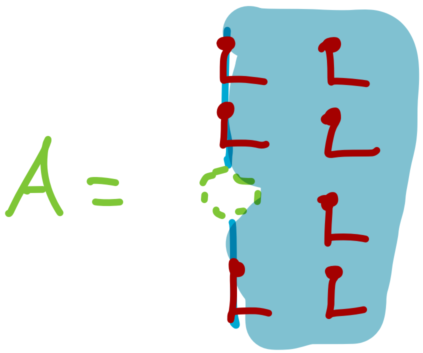
In this world, there is no "algebra of operators."
Even if you can continuously move operators near each other, the limit has no reason to exist.

Let's fix that. The derivative of translation is called the energy-momentum operator $P_\mu$.
Its spectrum is in the $\mathbb{R}^2$ momentum space dual to $\mathbb{R}^2$ spacetime.

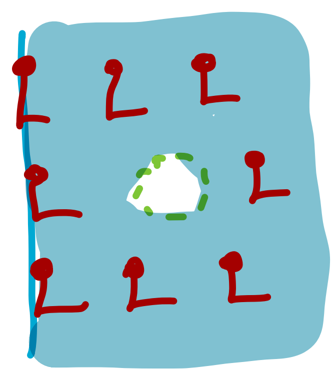
- Deep IR: scale $P_\mu \longrightarrow \infty$.
  Only $\mathrm{Ker}(P_\mu)$ remains. Look from far away.
- Deep UV: scale $P_\mu \longrightarrow 0$.
  Can only study pictures that fit on the slide of your microscope.

## Still get an algebra.

In the deep UV, translation is trivial.

So  is an algebra.

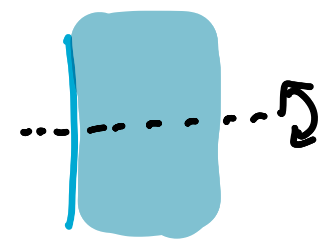
Nothing else exists. There isn't room for framings to rotate.

I think that remote detectability probably implies that $Z(A) =$ 

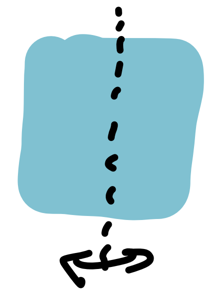
and that's basically it.

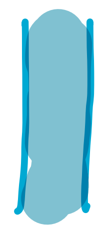
# 4. Flat QFTs with two reflections

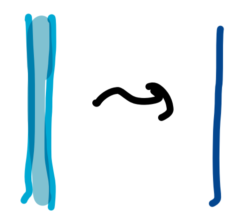
If your QFT admits a reflection symmetry normal to the boundary, then, as before, $A$ is a $*$-algebra.

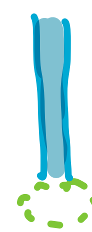


Suppose there is also a reflection symmetry parallel to the boundary.



Then we can form a strip: 

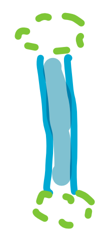
Since $P_\mu = 0$, can narrow the strip:  .

Now can form an interesting "space of states" $\mathcal{H} :=$  $=$  .

# Reflection positivity, the most important axiom.

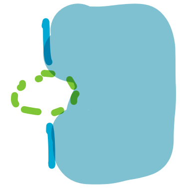
Our universe is defined over $\mathbb{R}$, or maybe over $\mathbb{C}$ w/ reflections acting antilinearly.

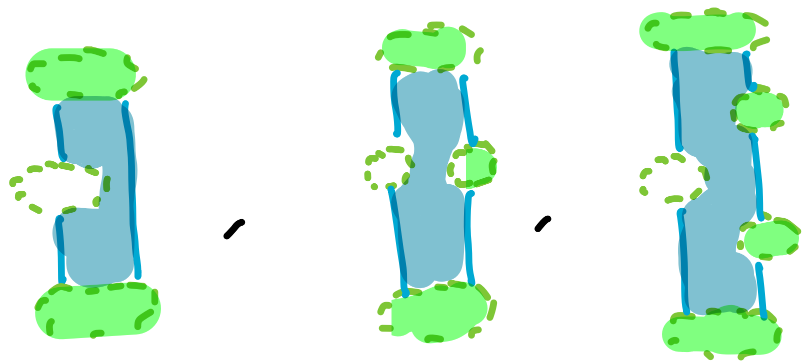
**Reflection positivity axiom:** If a closed diagram admits a reflection symmetry, then its value is positive.

E.g. The pairing  : $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ or $\mathbb{C}$ is positive and hence Hilbert by rem. det.

**Remark:** $\mathbb{R}$ has a topology. So any dual vector space has a topology, and remote detectability is really about continuous duals.

# The $W^*$ axiom

The algebra $A =$  can be closed to a closed diagram in lots of ways:

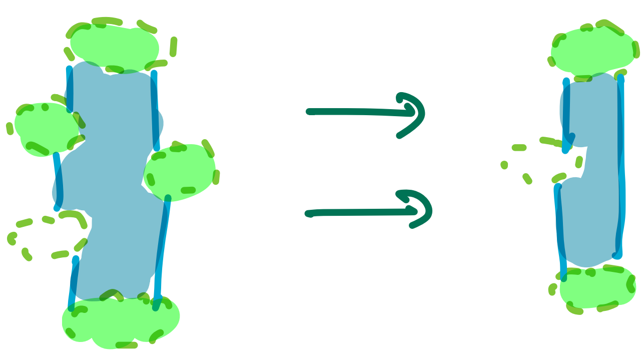
 , ...

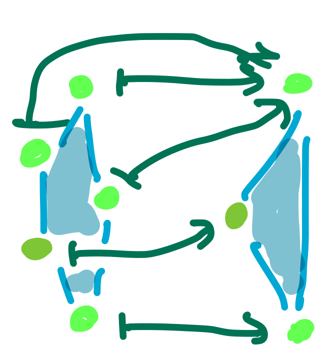
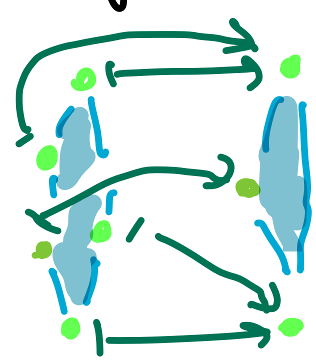
"  " means to fill it by any vector in the corresp. space.

This gives maps

$$A \to (H \otimes H)^\vee, \quad A \to (H \otimes A \otimes H)^\vee, \quad \ldots$$

Moreover, these fillings are related by fusion:



 , 

# The $W^*$ axiom

These assemble into a single map

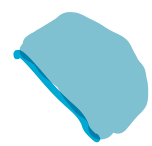
$$A \longrightarrow \lim\left(\text{some big diagram}\right)^\vee$$

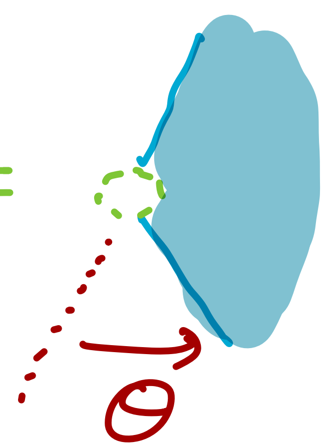
This limit computes $\mathrm{End}_{\mathrm{RMod}(A)}(H_A)$.

Remote Detectability says that this map is an isomorphism.

Since $H$ is Hilbert, this realizes $A$ as a von Neumann algebra!

# 5. Flat QFTs with O(2) symmetry

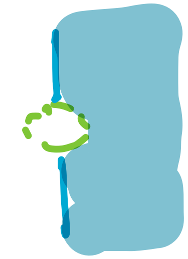
Now suppose our QFT is fully (Wick-rotated) special-relativistic. Then we can place boundaries at any angle: . The rule remains: near boundaries, local structure is product; boundaries cannot bend. But we can inquire about spaces of operators at corners.

$A_\theta =$ 

.

They are A-A bimodules.

# Remote detectability changes

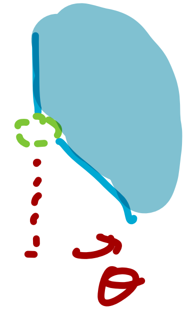
Now there are many more closures of .

So now remote detectability says

$$A \xrightarrow{\cong} \mathrm{Lim}\left(\text{some } \underline{\text{much bigger}} \text{ diagram}\right)^{\vee}$$

I don't know what this diagram computes in general.

# But von Neumann algebras still work

**Idea (Segal):** Assign $A_\theta =$  $= L^{2\pi/\theta}(A)$.

**Theorem (Pavlov):** This assignment satisfies remote detectability!!

In other words, von Neumann algebras have a secret $O(2)$ symmetry.

# Mysteries

- What is the "$O(2)$ action" on algebras that these are fixed points of?
- $L^p(A)$ is not invertible as a bimodule: only "part" of $O(2)$ acts. How to formulate this?
- What in physics rules out concave angles? Or are there "$L^{-p}(A)$" spaces to be investigated?
- Are there any other reflection positive remote-detectable 2D QFTs w/ boundary?